INDIFFERENCE PRICING OF THE EXPONENTIAL LEVY MODELS

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Abstract. We consider the geometric Levy processes and we study the utility indifference pricing approach for the European type option. Describing the investor's risk preferences by the so-called HARA-utilities we define the formulas for their value functions on the initially enlarged filtration and the equations for the indifference prices.

Key words and phrases: utility maximisation, utility indifference price, f-divergence minimal martingale measure, exponential Levy model

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1. Introduction

The geometric Levy models have been widely used since the 1990's to represent the asset prices. In comparison with the classical geometric Brownian motion modelling, which became the basis for the Black-Scholes option pricing theory, the use of the geometric Levy processes in asset price modelling has undisputed advantages. In the classical modelling, the log-returns produced by the geometric Brownian motion processes are normally distributed random variables which are far from the realistic for most time series of the financial data. The class of the geometric Levy models is also flexible enough to allow the process with either finite or infinite variation and finite or infinite activity on the market and, in particular, contains the classical Black-Scholes model (in the case of the a.s. continuous trajectories) and a number of the popular jump models [13].

It is well known that the exponential Levy models are in general incomplete and the problem of the option pricing reduces to the problem of the choice of

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suitable equivalent martingale measure. However, the unified algorithm for choosing such a suitable measure has not been emerged. From the other side, the assumption that one can always trade the asset is often rather restrictive because of low liquidity or legal restrictions. In some situations one can at best trade the asset correlated with some non-traded asset and at the same time to have an option on this non-traded asset. In these situations the traditional contingent claim valuation approaches are not applicable to the non-hedgeable contingent claims in the sense that one cannot construct the investment strategies that minimise the risk for the non-traded asset.

Nowadays, the utility indifference pricing becomes the main tool for the valuation of the claims in incomplete markets. Under the buyer's (seller's) indifference price of the claim one understands a price which an investor is willing to pay (to get) now to receive (to deliver) a claim at terminal time and to be indifferent to the situation of the non-having the claim, in the sense that his expected utility will not change under the optimal trading strategy in the both situations. The utility indifference pricing approach allows to hedge a risk from trading contingent claims and can be applied to the non-traded assets as well as to the traded asset. First, Hodges and Neurberges [22] asked at what price the investor is indifferent about a given claim under the transaction costs which bring the incompleteness on the market. Later this approach were extended in a number of papers, see for instance [3], [4], [6], [10], [34]. However, the explicit formulas of the indifference prices were derived only for Black-Scholes models ([32], [31]) and diffusion models ([20]), where the incompleteness on the market comes from the non-traded asset, for the stochastic volatility models ([19]), and in the situation of the complete market modelled by the by general semimartingales ([1]).

In this note we study the buyer's (seller's) indifference pricing of the European type option in the situation when one trades the asset on the finite time interval and at the same time has the European type option on the independent non-traded asset under the assumption that investor possesses some strong insider information about the non-traded asset. We consider the market model on the initially enlarged reference filtration which supports a traded asset and independent non-traded asset, both assets prices driven by the geometric Levy processes. The phenomenon of the enlarged filtration leads to the change of the set of self-financing and admissible strategies. In general, the reference filtration of the traded asset prices can be enlarged in the initial and progressive ways (see [24]).

More precisely, let T be a a fixed time horizon and we assume that trading is carried out on the time interval $[0,T]$. Let Levy process L be a process which generates the reference filtration and Levy process $\tilde{L}$ be an auxiliary process independent from L . We consider the market with the risk-less bond which is identically equal to 1, one traded risky-asset S and one non-traded risky-asset $\tilde{S}$ defined as

$$S_t = \mathcal{E}(L)_t, \quad \tilde{S}_t = \mathcal{E}(\tilde{L})_t, \quad t \in [0,T]$$

where $\mathcal{E}$ is a Doleans-Dade exponential. According to [23], the processes S and $\tilde{S}$ are semimartingales. To ensure that the price processes $S > 0$ and $\tilde{S} > 0$ we assume that the jumps $\Delta L > -1$ and $\Delta \tilde{L} > -1$. We consider the situation when investor is buying or selling the European-type option at time T. Let us denote by ξ the random variable which takes values on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and by μ its law. We will interpret the random variable ξ as a spot price of the non-traded underlying asset $\tilde{S}_T$ and the function $G_T = g(\xi)$, where g is a positive Borel function as a payoff of European type option. Then, considering that investor knows from the beginning the spot price at time T as a random variable, the new information flow is described, therefore, by the initially enlarged filtration.

Let x be an initial endowment and U be a utility function describing the investor's risk preferences. By $V(x,g)$ we denote the value function for the utility maximisation problem on the initially enlarged filtration. Then $V(x,g)$ is defined as

$$V(x,g) = \sup_{\varphi \in \Pi} \mathbb{E}\left(U\left(x + \int_0^T \varphi_s(\xi) dS_s + g(\xi) \right) \right),$$

where Π is a set of self-financing and admissible strategies on the initially enlarged filtration. Under the buyer's and seller's indifference prices we understand the quantities p_T^b and p_T^s which are the solution to the following equations

$$V(x,0) = V(x - p_T^b, g) \tag{1}$$

and

$$V(x,0) = V(x + p_T^s, -g) \tag{2}$$

respectively (for the details see [6], [32]). We remark that a following relation holds between p_T^b and p_T^s:

$$p_T^b(g) = -p_T^s(-g).$$

As we can see from (1) and (2) the indifference prices are the solutions to the equations consisting of two maximised utility function on the enlarged filtration. In our analysis we consider that the risk preferences of the investor are described by hyperbolic absolute risk (HARA) utilitiy functions.

There are two possible methods to evaluate the value function $V(x,g)$ and to solve the utility maximisation problem. The primal approach is based on the dynamic programming approach which leads to HJB equations for the value functions. However, in the most cases of the incomplete markets the associated HJB equations become the non-linear PDEs (see [20], [32]). We prefer to use the another approach which consists in the converting of the primal approach into the dual one which involves the minimisation problem over all equivalent martingale measures. This problem becomes a problem of finding the so-called f-divergence minimal equivalent measures if such a measure exists (see for instance [9],[7],[8],[10],[20],[30]).

The note is organised in a following way. Section 2 is devoted to the dual methods to solve the utility maximisation problem on the reference filtration and on the initially enlarged filtration. In Section 3 we discuss about the f-divergences minimal martingale measures on the both filtrations. In the cases of the exponential, logarithmic and power utilities we provide the explicit formulas for the Girsanov parameters of the changing of probability measure to the f-divergence minimal equivalent martingale measures. In Section 4 we present the main result for the utility maximisation problem on the initially enlarged filtration and the equations for the buyer's and seller's utility indifference prices for the exponential Levy models as well as their properties as risk measures. Section 5 is devoted to the calculus of the buyer's indifference prices in the setting of the defaultable geometric brownian motion model.

2. Methods to utility maximisation problem

In this section we introduce some known and new results concerning the maximising expected utility theory. We start by describing in detail our model. Let T be a fixed time horizon. Let L be a Levy process starting from 0 with generating triplets (b, σ^2, ν), where b is a drift parameters, σ^2 is a diffusion parameter and ν is a Levy measures, i.e. the measure on $\mathbb{R}$ satisfies the following two assumptions

$$\nu(\{0\}) = 0 \tag{3}$$

and

$$\int_{\mathbb{R}} \min\left\{x^2, 1\right\} \nu(dx) < \infty. \tag{4}$$

We remind that according to the Levy-Ito decomposition theorem a Levy process L with the generating triplet (b, σ^2, ν) can be represented in the following form: for $\forall t \in [0,T]$

$$L_t = \sigma W_t + bt + \int_0^t \int_{|x|>1} xN(dsdx) + \int_0^t \int_{|x|\leq 1} x\tilde{N}(dsdx), \tag{5}$$

where W is a Winer process, $N(ds, dx)$ is a Poisson random measure and $\tilde{N}(ds, dx)$ is a compensated Poisson random measure with a compensator $\nu(dx)ds$.

We consider a canonical probability space $(\Omega, \mathcal{F}_T, P)$ endowed with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by process $L = (L_t)_{0\leq t\leq T}$. We suppose that financial market consists of the risk-less bond which is identically equal to 1, one traded risky-asset $S = \mathcal{E}(L)$ and one non-traded risky-asset $\tilde{S} = \mathcal{E}(\tilde{L})$, with $S_0 = 1$ and $\tilde{S}_0 = 1$, where $\mathcal{E}(\cdot)$ is a stochastic exponential,

$$\mathcal{E}(L)_t = \exp\left\{L_t - \frac{1}{2} < L >_t\right\} \prod_{0\leq s\leq t} \exp\{-\Delta L_s\}(1 + \Delta L_s).$$

According to Theorem 4.61 in [23], the process S is a semimartingale. To ensure that the price process $S > 0$ we assume that the jumps $\Delta L > -1$. By Lemma A.8 in [17], the assumption that $\Delta L > -1$ is in fact equivalent to the existence of a Levy process $L^{'}$ such that $S_t = \mathcal{E}(L)_t = \exp(L^{'}_t)$. Lemma A.8 in [17] also gives the following formulae to determine the Levy triplet $(b^{'}, \sigma^{'2}, \nu^{'})$ of $L^{'}$:

$$\begin{aligned}
\sigma^{'} &= \sigma, \\
b^{'} &= b - \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (h(\ln(1 + x)) - h(x))\nu(dx), \\
\nu^{'}(G) &= \int_{\mathbb{R}} 1_G(\ln(1 + x))\nu(dx), \text{ for } \forall G \in \mathcal{B}(\mathbb{R}),
\end{aligned}$$

where h is a bounded truncation function. In fact, if we apply Ito's formula to the function $\ln S_T = L^{'}_T$ we will see, that for an existence of the characteristics $(b^{'}, \sigma^{'2}, \nu^{'})$ of $L^{'}$ it is sufficient to define the truncation function in a specific way, for example:

$$h(x) = x 1_{|x|\leq\frac{1}{2}}(x).$$

We remark that the non-traded risky assets $\tilde{S}$ also satisfies all the above properties.

By means of the relation between L and $L^{'}$, it was shown in [21] that a Levy process is a local martingale if and only if its stochastic exponential is a local martingale. For this reason we prefer to model the asset's price dynamic by the stochastic exponential Levy processes.

We assume that at time T, investor is buying or selling the European-type option with payoff function G_T and the underlying non-traded asset's price is driven by the geometric Levy process $\tilde{S}$. Let us denote by ξ the random variable which takes values on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and by μ its law. We will interpret the random variable ξ as a spot price $\tilde{S}_T$. Then, the payoff function of European type option $G_T = g(\xi)$, where g is a positive Borel function.

We assume that investor possesses the strong insider information about the spot price of the non-traded asset, i.e. knows ξ as a random variable from the beginning of the trading. Then, on the product space $(\Omega \times \mathbb{R}, \mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}))$ we define a probability $\mathbb{P}$ as it follows: for all $A \in \mathcal{F}$ and $B \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}(A \times B) = \int_B P(A) d\mu(u), \tag{6}$$

such that $\mathbb{P}(A \times \mathbb{R}) = P(A)$ and $\mathbb{P}(\Omega \times B) = \mu(B)$.

Now we define the initially enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]}$ by

$$\mathcal{G}_t = \bigcap_{s>t} \left(\mathcal{F}_s \otimes \sigma(\xi)\right). \tag{7}$$

The set of admissible and self-financing strategies, being the set of predictable functions, depend highly on the probability space and the filtration which we consider. We denote by $\Pi(\mathbb{F})$ the set of self-financing and admissible strategies φ on the reference filtration, i.e. $\mathbb{F}$-predictable, S-integrable on $[0,T]$ investment strategies with the wealth bounded from below P-a.s.. By $\mathcal{P}(\mathbb{F})$ we denote $\mathbb{F}$-predictable processes, then the set $\Pi(\mathbb{F})$ is of the form

$$\Pi(\mathbb{F}) = \left\{\varphi \in \mathcal{P}(\mathbb{F}) \mid \exists c \in \mathbb{R}_+ : (\varphi \cdot S)_t \geq -c \ \forall t \in [0,T] \ \ P- \text{ a.e.}\right\},$$

where $(\varphi \cdot S)_t = \int_0^T \varphi_s dS_s$ is a stochastic integral. (For definition of a stochastic integral see [23].)

Next, we denote by $\Pi = \Pi(\mathbb{G})$ the set of investment strategies $\varphi(\xi)$ on the initially enlarged filtration such that $\varphi(\xi)$ is $\mathbb{G}$-predictable and S-integrable on $[0,T]$ with the wealth $\mathbb{P}$-a.s. bounded from below. In next lemma we recall the known result about the $\mathbb{G}$-predictable processes.

Lemma 1. (*cf.* [5]) *A random process $\varphi(\xi)$ is $\mathbb{G}$-predictable if and only if the application $(t, \omega, \xi) \to \varphi(\xi)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable random variable.*

Hence, the set of the self-financing and admissible strategies on the initially enlarged filtration is of the form

$$\Pi(\mathbb{G}) = \left\{\varphi(\xi) \in \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \mid \exists c \in \mathbb{R}_+ : (\varphi(\xi) \cdot S)_t \geq -c \ \forall t \in [0,T] \ \ \mathbb{P}- \text{ a.e.}\right\}.$$

In order to find the indifference prices of the claim in the considered model we solve two utility maximisation problems in (1) and (2). We start from the solution to the classical utility maximisation problem.

2.1. **Classical utility maximisation problem.** Again $(\Omega, \mathcal{F}_T, \mathbb{F}, P)$ is a filtered probability space. By $\mathbb{E}$ we denote the mathematical expectation with respect to probability measure $\mathbb{P}$ and by E_P, E_μ, the mathematical expectations with respect to probability measure P and to measure μ respectively. Under the classical utility maximisation problem we imply throughout of the note the optimisation problem of the form

$$\bar{V}(x,0) = \sup_{\varphi \in \Pi(\mathbb{F})} E_P \left[U \left(x + \int_0^T \varphi_s dS_s \right) \right].$$

Similarly as in [9], [18] we consider a utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, which is assumed to be strictly increasing, strictly concave, continuously differentiable in $dom(U) = \{x \in \mathbb{R} | U(x) > -\infty\}$ and to satisfy the Inada conditions

$$U^{'}(\infty) = \lim_{x \to \infty} U^{'}(x) = 0,$$
$$U^{'}(\overline{x}) = \lim_{x \downarrow \underline{x}} U^{'}(x) = +\infty,$$

for $\underline{x} = \inf\{x \in \mathbb{R} | U(x) > -\infty\}$.

We require that the utility function is the increasing function of the wealth because the growth of wealth the investor's usefulness also grows. The concavity of the function stands for the risk-aversion of the investor.

In the next proposition we present the auxiliary result concerning the solution to the utility maximisation problem on the reference and on the initially enlarged filtration.

Proposition 1. *Let $\Pi(\mathbb{F})$ and $\Pi(\mathbb{G})$ be the sets of the self-financing and admissible strategies on the reference filtration and initially enlarged filtration respectively. Then,*

$$\sup_{\varphi \in \Pi(\mathbb{F})} E_P \left[U \left(x + \int_0^T \varphi_s dS_s \right) \right] = \sup_{\varphi(\xi) \in \Pi(\mathbb{G})} \mathbb{E} \left[U \left(x + \int_0^T \varphi_s(\xi) dS_s \right) \right]. \tag{8}$$

Proof:

Let us set

$$\Pi^u(\mathbb{F}) = \left\{\varphi(u) \in \mathcal{S}_u\left(\mathcal{P}\left(\mathbb{F}\right) \otimes \mathcal{B}(\mathbb{R})\right) \mid \exists c \in \mathbb{R}_+ : (\varphi(\xi) \cdot S)_t \geq -c \, \forall t \in [0,T] \; \mathbb{P}\text{– a.e..}\right\},$$

where $\mathcal{S}_u\left(\mathcal{P}\left(\mathbb{F}\right) \otimes \mathcal{B}(\mathbb{R})\right)$ is a section of $\mathcal{P}\left(\mathbb{F}\right) \otimes \mathcal{B}(\mathbb{R})$ in u.

Then, if $\varphi(u) \in \Pi^u(\mathbb{F})$ then $\varphi(u) \in \Pi(\mathbb{F})$. From the independence of L and ξ and the fact that $U\left(x + \int_0^t \varphi_s(\xi) S_s\right)$ is a $\mathcal{B}(\mathbb{R})$- measurable function of $(\varphi \cdot S_T)$ for all $x \in \mathbb{R}_+$, it follows that

$$\mathbb{E}\left[U\left(x + \int_0^t \varphi_s(\xi) dS_s\right) \Big| \xi = u\right] = E_P\left[U\left(x + \int_0^t \varphi_s(\cdot, u) S_s\right)\right].$$

Then, one gets:

$$\begin{aligned}
\mathbb{E}\left[U\left(x + \int_0^t \varphi_s(\xi) S_s\right)\right] &= \int_{\mathbb{R}} \mathbb{E}\left[U\left(x + \int_0^T \varphi_s(u) S_s\right)\right] d\mu(u) \\
&\leq \int_{\mathbb{R}} \sup_{\varphi(u) \in \Pi^u(\mathbb{F})} E_P\left[U\left(x + \int_0^T \varphi_s(u) S_s\right)\right] d\mu(u) \\
&\leq \int_{\mathbb{R}} \sup_{\varphi \in \Pi(\mathbb{F})} E_P\left[U\left(x + \int_0^T \varphi_s S_s\right)\right] d\mu(u) \\
&= \sup_{\varphi \in \Pi(\mathbb{F})} E_P\left[U\left(x + \int_0^T \varphi_s dS_s\right)\right],
\end{aligned}$$

and, hence

$$\sup_{\varphi \in \Pi(\mathbb{F})} E_P\left[U\left(x + \int_0^T \varphi_s dS_s\right)\right] \geq \sup_{\varphi(\xi) \in \Pi(\mathbb{G})} \mathbb{E}\left[U\left(x + \int_0^T \varphi_s(\xi) dS_s\right)\right].$$

The inequality in the opposite sense follows from the fact that each $\varphi \in \Pi(\mathbb{F})$ can be extended on $\Pi(\mathbb{G})$ as $\varphi(t, \omega, \xi) = \varphi(t, \omega)$, for $\forall t \in [0,T]$, $\forall \omega \in \Omega$ and $\forall \xi \in \mathbb{R}$. Finally, we have (8).

$\square$

In the next discussions concerning the classical utility maximisation theory we will use the notation $V(x,0)$ for the classical utility maximisation problem instead of $\bar{V}(x,0)$ since we have proved that $\bar{V}(x,0) = V(x,0)$.

2.2. **Dual approach to utility maximisation problem.** The dual approach to the classical utility maximisation problem consists in the converting of the primal problem into the dual one which involves the minimisation problem over all equivalent martingale measures on the reference filtration.

More precisely, dual optimisation problem is derived in the following way. (For details see [18], [20]). Let $\mathcal{M}(\mathbb{F})$ be the set of locally equivalent to P martingale measures. Let us denote by I the inverse function of $U^{'}$, i.e.

$$I = \left(U^{'}\right)^{-1}$$

and let f be the convex conjugate function of the utility function U, namely

$$f(y) = \sup_{x>0} \left\{U(x) - yx\right\} = U(I(y)) - yI(y). \tag{9}$$

The convex function f is a Frenchel-Legendre transform of U and from equation (9) the relation $I(y) = -f^{'}(y)$ follows. The duality here can be explained by the fact that U is also a Frenchel-Legendre transform of f such that

$$U(x) = \sup_{y>0} \left\{f(y) + xy\right\} = f(I(x)) + xI(x).$$

In this paper we consider the HARA utilities. They are the utilities U such that its convex conjugate f verifies:

$$f^{''}(x) = ax^{\gamma},$$

with $a > 0$ and $\gamma \in \mathbb{R}$. As known, such functions f can be represented in the form

$$f(x) = Af_{\gamma}(x) + Bx + C,$$

with $A > 0$, $B, C \in \mathbb{R}$ and the function $f_{\gamma} = f_{\gamma}(x)$ given by

$$f_{\gamma}(x) = \begin{cases} c_{\gamma}x^{\gamma+2}, & \text{if } \gamma \neq -1, -2, \\ x \log x, & \text{if } \gamma = -1, \\ -\log x, & \text{if } \gamma = -2. \end{cases}$$

The functions from up to down correspond to the power, exponential and logarithmic utilities respectively. We recall the following correspondences between utility functions U and f-divergence functions:

$$\begin{aligned} \text{if } U(x) &= \log x, \text{ then } f(x) = -\ln(x) - 1, \\ \text{if } U(x) &= \frac{x^p}{p},\ p < 1, \text{ then } f(x) = -\frac{p-1}{p}x^{\frac{p}{p-1}}, \\ \text{if } U(x) &= 1 - e^{-\gamma x},\ \gamma > 0,\ \text{then } f(x) = 1 - \frac{x}{\gamma} + \frac{1}{\gamma}x\ln x - \frac{1}{\gamma}x\ln\gamma. \end{aligned}$$

Definition 1. *The f-divergence minimal equivalent martingale measure on $[0,T]$ is equivalent to P_T probability measure Q^*_T on $\mathcal{F}_T$ which satisfies the following equality:*

$$E_P\left[f\left(\frac{dQ^*_T}{dP_T}\right)\right]=\inf_{Q\in\mathcal{M}(\mathbb{F})}\left\{E_P\left[f\left(\frac{dQ_T}{dP_T}\right)\right]\right\}.$$

Definition 2. *We say that an f-divergence minimal martingale measure Q^* on $\mathcal{F}_T$ is invariant under scaling if for all $c\in\mathbb{R}_+$*

$$E_P\left[f\left(c\frac{dQ^*_T}{dP_T}\right)\right]=\inf_{Q\in\mathcal{M}(\mathbb{F})}\left\{E_P\left[f\left(c\frac{dQ_T}{dP_T}\right)\right]\right\}.$$

Let us denote by X_T the wealth at time T, i.e. $X_T=x+\int_0^T\varphi_s dS_s$, where x is the initial capital and $\varphi\in\Pi(\mathbb{F})$. For any equivalent probability measure $Q\in\mathcal{M}(\mathbb{F})$ we introduce the following maximisation problem

$$V_Q(x)=\sup_{X_T}\left\{E_P(U(X_T)):X_T\in L^1(Q),\ E_Q(X_T)\le x,\ E_P(U(X_T)^-)<\infty\right\}.$$

Next we recall the well-known result of Goll and Ruschendorf which gives us the dual representation of $V(x,0)$.

Proposition 2. (*cf.* [18], *Lemma 4.1*) *Let Q be a probability measure dominated by P and $E_Q\left(-f'\left(\lambda\frac{dQ|_T}{dP|_T}\right)\right)<\infty$ for $\forall\lambda\in\mathbb{R}_+$. Then*

(i) $V_Q(x)=\inf_{\lambda>0}\left\{E_P\left(f\left(\lambda\frac{dQ|_T}{dP|_T}\right)\right)+\lambda x\right\}$.

(ii) There is a unique solution for λ to the equation $E_Q\left(-f'\left(\lambda\frac{dQ|_T}{dP|_T}\right)\right)=x$, denoted as $\lambda_Q(x)\in\mathbb{R}_+$, and $V_Q(x)=E_P\left[U\left(-f'\left(\lambda_Q(x)\frac{dQ|_T}{dP|_T}\right)\right)\right]$.

Therefore, the similar result holds for the f-divergence minimal martingale measure Q^* and

$$V_{Q^*}(x)=\inf_{\lambda>0}\inf_{Q\in\mathcal{M}(\mathbb{F})}\left\{E_P\left(f\left(\lambda\frac{dQ|_T}{dP|_T}\right)\right)+\lambda x\right\}. \tag{10}$$

2.3. **Dual approach with initially enlarged filtration.** Let $(\Omega\times\mathbb{R},\mathcal{F}_T\otimes\mathcal{B}(\mathbb{R}),\mathbb{G},\mathbb{P})$ be a filtered probability space, where $\mathbb{G}$ and $\mathbb{P}$ are defined in (6) and (7) respectively. We remind that the utility maximisation problem on the initially enlarged filtration is of the form:

$$V(x,g) = \sup_{\varphi \in \Pi(\mathbb{G})} \mathbb{E}\left(U\left(x + \int_0^T \varphi_s(\xi) dS_s + g(\xi)\right)\right). \tag{11}$$

The set of the admissible and the self-financing portfolios depends highly on the filtration which we considered. In the framework of our model, the set of such portfolios on the initially enlarged filtration is represented by the set $\Pi(\mathbb{G})$.

In the terms of the dual approach, the set of the equivalent martingale measures will be changed when we replace the filtration of the price process by the initially enlarged filtration. Let $\mathcal{M}(\mathbb{G})$ be a set of $\mathbb{P}$-equivalent martingale measures on the initially enlarged filtration $\mathbb{G}$ defined as

$$\mathcal{M}(\mathbb{G}) = \left\{\mathbb{Q} : \ \mathbb{Q} \overset{loc}{\sim} \mathbb{P} \text{ and } S \text{ is } (\mathbb{Q},\mathbb{G})\text{-martingale}\right\}.$$

Then the restrictions of the measures $\mathbb{Q}$ on the σ-algebra $\mathcal{G}_T$ can be given by

$$\frac{d\mathbb{Q}|_{\mathcal{G}_T}}{d\mathbb{P}|_{\mathcal{G}_T}} = Z_T(\xi), \tag{12}$$

where the process $Z(\xi) = (Z_t(\xi))_{t\in[0,T]}$ is a uniformly integrable $(\mathbb{P},\mathbb{G})$-martingale and the random variable $Z_T(\xi)$ has to be positive random variable with

$$\mathbb{E}\left[Z_T(\xi)\right] = 1.$$

Since $Z_0(\xi)$ is not necessary equal to 1, we introduce a process $\tilde{Z}(\xi) = \left(\tilde{Z}_t(\xi)\right)_{t\in[0,T]}$ such that for $t > 0$, $\tilde{Z}_t(\xi) = \frac{Z_t(\xi)}{Z_0(\xi)}$ and

$$\tilde{Z}_0(\xi) = \frac{Z_0(\xi)}{E_\mu Z_0(\xi)}.$$

The process $\tilde{Z}(u)$ is a positive $(P,\mathbb{F})$-martingale (cf. [5], Proposition 2.1) and, moreover, is a density process of the equivalent martingale measures Q^u such that $\frac{dQ^u|_{\mathcal{F}_T}}{dP|_{\mathcal{F}_T}} = \tilde{Z}_T(u)$. For $u \in supp(\mu)$ we define the set of the equivalent to P martingale measures Q^u such that

$$\mathcal{M}^u(\mathbb{G}) = \left\{Q^u : Q^u \overset{loc}{\sim} P, \ S \text{ is } (Q^u,\mathbb{F})\text{-martingale and } \mathbb{Q} \in \mathcal{M}(\mathbb{G})\right\}.$$

Let us denote by $\mathcal{K}$ the set of the measures $\mathbb{Q} \in \mathcal{M}(\mathbb{G})$ such that the corresponding f- divergence function is $\mathbb{P}$-integrable, i.e.

$$\mathcal{K} = \Big\{ \mathbb{Q} \in \mathcal{M}(\mathbb{G}) : \quad \mathbb{E} \Big| f \left(\frac{d\mathbb{Q}|_{\mathcal{G}_T}}{d\mathbb{P}|_{\mathcal{G}_T}} \right) \Big| < \infty \Big\}.$$

Next, let us put

$$\tilde{\mathcal{K}} = \Big\{ \mathbb{Q} \in \mathcal{K} : \quad E_\mu Z_0(\xi) = 1 \Big\}.$$

In the next proposition we give the dual representation of the value function $V(x, g)$ on the initially enlarged filtration.

Proposition 3. *If the f-minimal equivalent martingale measure $\mathbb{Q}^*$ exists and belongs to $\tilde{\mathcal{K}} \neq \emptyset$, then the utility optimisation problem on initially enlarged filtration can be written in a following dual form: for $\lambda > 0$*

$$V(x, g) = \inf_{\lambda > 0} \inf_{\mathbb{Q} \in \tilde{\mathcal{K}}} \left\{ \mathbb{E} \left[f \left(\lambda \frac{d\mathbb{Q}|_T}{d\mathbb{P}|_T} \right) + \lambda Z_0(\xi)(x + g(\xi)) \right] \right\}. \tag{13}$$

Proof:

Let $\varphi(\xi) \in \Pi(\mathbb{G})$. Then, from the independence of L and ξ and the fact that $U\left(x + \int_0^t \varphi_s(\xi) S_s + g(\xi)\right)$ is a $\mathcal{B}(\mathbb{R}^2)$- measurable function of $(\varphi \cdot S_T, g)$ for all $x \in \mathbb{R}_+$, it follows that

$$\mathbb{E} \left[U \left(x + \int_0^t \varphi_s(\xi) dS_s + g(\xi) \right) \Big| \xi = u \right] = E_P \left[U \left(x + \int_0^t \varphi_s(\cdot, u) S_s + g(u) \right) \right].$$

The strategies $\varphi(\omega, u) \in \Pi^u(\mathbb{F})$ and $u \in supp(\mu)$. Let us define the wealth process $(X_t(u))_{t \in [0,T]}$ as

$$X_t(u) = x + g(u) + \int_0^t \varphi_s(u) dS_s,$$

where $\varphi(u) \in \Pi^u(\mathbb{F})$. Hence, the conditional utility maximisation problem is of the form

$$V^u(x, g) = \sup_{X_T(u)} E_P \left[U \left(X_T(u) \right) \right].$$

Since the terminal wealth is generated from the initial wealth $x + g(\xi)$ using a self-financing strategy, the self -financing condition becomes

$$\mathbb{E} \left[Z_T(\xi) X_T \big| \xi = u \right] \leq \mathbb{E} \left[Z_T(\xi)(x + g(\xi)) \big| \xi = u \right],$$

or using that $Z_T(u) = Z_0(u)\tilde{Z}_T(u)$ we can write the self-financing condition in a following form:

$$E_P\left[Z_0(u)\tilde{Z}_T(u)X_T(u)\right] \leq Z_0(u)\left(x + g(u)\right). \tag{14}$$

For the equivalent probability measures $Q^u \in \mathcal{M}^u(\mathbb{G})$ we introduce the following maximisation problem

$$V^u_{Q^u}(x) = \sup_{X_T}\Big\{E_P(U(X_T)) : X_T \in L^1(Q^u),$$
$$E_{Q^u}\left[Z_0(u)(X_T(u))\right] \leq Z_0(u)\left(x + g(u)\right),\ E_P(U(X_T)^-) < \infty\Big\}.$$

Let us set $\tilde{\lambda}(u) = \lambda Z_0(u)$ and $\tilde{\lambda}(u) > 0$. Then, using (i) from Proposition 2 we get that

$$V^u_{Q^u}(x) = \inf_{\tilde{\lambda}(u)>0}\left\{E_P\left(f\left(\tilde{\lambda}(u)\frac{dQ^u|_T}{dP|_T}\right)\right) + \tilde{\lambda}(u)x\right\}.$$

In particular, for $Q^{*,u}$ these formulas are true and from (ii) of Proposition 2, it follows that there is a unique solution for $\tilde{\lambda}^*(u)$ to the equation

$$E^{*,u}_Q\left(-f'\left(\tilde{\lambda}^*(u)\frac{dQ^{*,u}|_T}{dP|_T}\right)\right) = x + g(u),$$

denoted as $\tilde{\lambda}_{Q^{*,u}}(x) \in \mathbb{R}_+$, and that

$$V^u_{Q^{*,u}}(x) = E_P\left[U\left(-f'\left(\tilde{\lambda}^*(u)_{Q^{*,u}}(x)\frac{dQ^{*,u}|_T}{dP|_T}\right)\right)\right].$$

Then, we can conclude that

$$\sup_{X_T(u)} E_P\left[U\left(X_T(u)\right)\right] = \inf_{\tilde{\lambda}>0}\inf_{\tilde{Z}_T(u)}\left\{E_P\left[f(\tilde{\lambda}(u)\tilde{Z}_T(u))\right] + \tilde{\lambda}(u)(x + g(u))\right\}. \tag{15}$$

Then, if the measure $Z^*_T(u)$ is the minimal divergence martingale measure, according to Theorem 3.1 in [18], we know that

$$-f'(\tilde{\lambda}(u)Z^*_T(u)) = x + \int_0^T \varphi^*_s(\cdot, u)dS_s + g(u)$$

for some process $\varphi^*(u) \in \Pi^u(\mathbb{F})$ such that $\varphi^*(u)\cdot S_T$ is martingale under Q^{u*}.

Then, according to (ii) of Theorem 5.1 in [18], we can conclude that

$$\sup_{\varphi\in\Pi^u(\mathbb{F})} E_P\left[U\left(x+\int_0^T \varphi_s(\cdot,u)dS_s+g(u)\right)\right]=E_P\left[U\left(x+\int_0^T \varphi^*_s(\cdot,u)dS_s+g(u)\right)\right]$$

and $\varphi^*(\xi)$ is an admissible optimal portfolio strategy.

Then, one gets that

(16)

$$E_P\left[\left(x+\int_0^T \varphi^*_s(\cdot,u)S_s+g(u)\right)\right]=E_P\left[f(\tilde{\lambda}^*(u)\tilde{Z}^*_T(u))\right]+\tilde{\lambda}^*(u)(x+g(u)),$$

where the corresponding minimal divergence measure is uniquely defined by the value of $\tilde{\lambda}^*(u)$.

Now, we prove that $V(x,g)=\int_{\mathbb{R}} V^u(x,g)d\mu(u)$. Taking into account the independence of L and ξ we get:

$$\begin{aligned}\mathbb{E}\left[U\left(x+\int_0^T \varphi_s(\xi)dS_s(\xi)+g\left(\xi\right)\right)\right] &= \int_{\mathbb{R}} E_P\left[U\left(x+\int_0^T \varphi_s(\cdot,u)dS_s+g(u)\right)\right]d\mu(u)\\ &\leq \int_{\mathbb{R}} \sup_{\varphi(u)\in\Pi^u(\mathbb{F})} E_P\left[U\left(x+\int_0^T \varphi_s(u)dS_s+g(u)\right)\right]d\mu(u).\end{aligned}$$

For each $\epsilon>0$ there exists $\varphi^{(\epsilon)}(u)\in\Pi^u(\mathbb{F})$ such that

$$\sup_{\varphi\in\Pi^u(\mathbb{F})} E_P\left[U\left(x+\int_0^T \varphi_s(u)dS_s+g(u)\right)\right]\leq E_P\left[U\left(x+\int_0^T \varphi^{(\epsilon)}_s(u)dS_s+g(u)\right)\right]+\epsilon.$$

Integration with respect to μ gives:

$$\begin{aligned}\int_{\mathbb{R}} & \sup_{\varphi\in\Pi^u(\mathbb{F})} E_P\left[U\left(x+\int_0^T \varphi_s(u)dS_s+g(u)\right)\right]d\mu(u)\\ \leq & \int_{\mathbb{R}} E_P\left[U\left(x+\int_0^T \varphi^{(\epsilon)}_s(u)dS_s+g(u)\right)\right]d\mu(u)+\epsilon\\ = & \mathbb{E}\left[U\left(x+\int_0^T \varphi^{(\epsilon)}(\xi)_s dS_s(\xi)+g(\xi)\right)\right]+\epsilon\\ \leq & V(x,g)+\epsilon.\end{aligned}$$

Combining the both inequalities we get :

$$V(x,g)=\int_{\mathbb{R}} E_P\left[\left(x+\int_0^t \varphi^*_s(\cdot,u)S_s+g(u)\right)\right]d\mu(u)$$

and from (16) we obtain the following equality:

$$V(x,g) = \int_{\mathbb{R}} \left(E_P \left[f(\tilde{\lambda}^*(u)\tilde{Z}_T^*(u)) \right] + \tilde{\lambda}^*(u)(x+g(u)) \right) d\mu(u),$$

where the corresponding minimal divergence measure is uniquely defined by the value of $\tilde{\lambda}^*(u)$.

Using the independence of ξ and L and the fact that $\tilde{\lambda}^*(u) = \lambda^* Z_0^*(u)$, $Z_T^*(u) = Z_0^*(u)\tilde{Z}_T^*(u)$, and that $Z_T(\xi)$ is the density process of the equivalent martingale measure defined on the enlarged filtration, we obtain the following relation:

$$\begin{aligned}
\int_{\mathbb{R}} \left(E_P \left[f(\tilde{\lambda}^*(u)\tilde{Z}_T^*(u)) \right] \right. & + \left. \tilde{\lambda}^*(u)(x+g(u)) \right) d\mu(u) \\
& = \int_{\mathbb{R}} \left(E_P \left[f(\lambda^* Z_T^*(u)) \right] + \lambda^* Z_0^*(u)(x+g(u)) \right) d\mu(u) \\
& = \inf_{Z_T(\xi)} \left\{ \int_{\mathbb{R}} \left(E_P \left[f(\lambda^* Z_T(u)) \right] + \lambda^* Z_0^*(u)(x+g(u)) \right) d\mu(u) \right\} \\
& = \inf_{\lambda>0} \inf_{\mathbb{Q}\in\tilde{\mathcal{K}}} \left\{ \mathbb{E} \left[f\left(\lambda \frac{d\mathbb{Q}|_T}{d\mathbb{P}|_T} \right) + \lambda Z_0(\xi)(x+g(\xi)) \right] \right\}.
\end{aligned}$$

Then, the equation (13) holds.

□

3. f-divergence minimal equivalent martingale measures for geometric Levy models

Proposition 2 and Proposition 3 establish that the dual problem becomes a problem of finding the so-called f-divergence minimal equivalent measures if such a measure exists. A general characterisation of f-divergence minimal martingale measure was given first in [18]. Then, for the semimartingale models the necessary and sufficient conditions of the existence and uniqueness the minimal equivalent measures were formulated first in [15] and the f-divergence minimal equivalent measures for the geometric Levy processes models were studied in the number of papers and books, see for instance [7], [8],[9], [16], [25], [30].

In this section we provide the explicit form of the Girsanov's parameters of the changing of measure P to the f-divergence minimal equivalent martingale measures on the reference filtration $\mathbb{F}$. Then we determine the density process of the f-divergence minimal martingale measure on the initially enlarged filtration and we find its expression through the density process of the f-minimal martingale measure on the reference filtration.

3.1. f-divergence minimal equivalent martingale measures on $\mathbb{F}$ and corresponding information processes.

The equivalent martingale measure $Q \in \mathcal{M}(\mathbb{F})$ can be given by its density process $Z = (Z_s)_{s\in[0,T]}$ which is a uniformly integrable $(\mathbb{F}, P)$-martingale. In turn, Z can be written in the form of $Z = \mathcal{E}(M)$, where M is a local martingale and $\mathcal{E}(\cdot)$ is a Doleans-Dade exponential. Let h be a truncation function. From Girsanov's theorem we know that there exists two predictable functions β and Y verifying the following conditions $\forall s \in [0, T]\,,\ P-$ a.s.

$$\int_0^t \beta_s^2 ds < \infty \text{ and } \int_0^t \int_{\mathbb{R}} h(x)(Y_s(x) - 1)\nu(dx)ds < \infty,$$

such that

$$M_t = \int_0^t \sigma\beta_s dW_s + \int_0^t \int_{\mathbb{R}} (Y_s(x) - 1)\,\tilde{N}(ds, dx). \tag{17}$$

We will refer to the pair (β, Y) as the Girsanov's parameters of the changing of measure from P to the equivalent martingale measure Q.

For the price process S to be a $(Q, \mathbb{F})$-martingale it is necessary and sufficient to have a new drift equal zero under the measure Q. Since $S = \mathcal{E}(L)$ it follows that process S is a martingale if and only if L is a martingale. Then it is enough that the new drift with respect to the measure Q of process L will be equal zero. From the Girsanov's theorem we know that the knew drift of Levy process L with the Levy triplet (b, σ^2, ν) with respect to the measure Q is of the form

$$B_t^Q = bt + \int_0^t \beta_s\sigma^2 ds + \int_0^t \int_{\mathbb{R}} h(x)(Y_s(x) - 1)d\nu(x).$$

It was shown in [11], [12] , [23], [30] that when P is a law of the Levy process L the Girsanov's parameters of minimal martingale measure are independent on (ω, t). Then, if there exists the minimal martingale measure Q^*, its Girsanov's parameters are $\beta \in \mathbb{R}$ and a positive measurable function Y such that for all $t \in [0, T]$ and all $\omega \in \Omega$, $\beta_t(\omega) = \beta$ and $Y_t(\omega, x) = Y(x)$. Then the condition for the process S to be a $(Q, \mathbb{F})$-martingale is of the form:

$$b + \beta\sigma^2 + \int_{\mathbb{R}} (Y(x) - 1)h(x)\nu(dx) = 0,\ P\text{-a.s.}. \tag{18}$$

We set

$$\bar{\mathcal{K}} = \left\{ (\beta, Y) : (18) \text{ is fulfilled, } E_P \left| f\left(Z_T^{(\beta,Y)} \right) \right| < \infty \right\},$$

and we reformulate Definition 1 of the f-divergences minimal martingale measures in the terms of the density processes.

Definition 3. *The f-divergence minimal equivalent martingale measure for the geometric Levy models is the equivalent probability measure $Q^* \in \mathcal{M}(\mathbb{F})$ which satisfies the following two conditions:*

(i) $\frac{dQ^*|_T}{dP|_T} = Z_T(\beta^*, Y^*)$*;*

(ii) $E_P\left[f\left(Z_T\left(\beta^*, Y^*\right)\right)\right] = \inf_{(\beta,Y)\in\bar{\mathcal{K}}} E_P\left[f\left(Z_T\left(\beta, Y\right)\right)\right].$

We introduce three important quantities related with P_T and Q_T^* namely the entropy of P with respect to Q_T^*,

$$\mathbf{I}(P_T|Q_T^*) = -E_P\left[\ln Z_T^*\right],$$

the entropy of Q_T^* with respect to P_T,

$$\mathbf{I}(Q_T^*|P_T) = E_P\left[Z_T^* \ln Z_T^*\right],$$

and Hellinger type integral

$$H_T^{(q*)} = E_P\left[(Z_T^*)^q\right], \quad q = \frac{p}{p-1}, \quad p < 1.$$

Proposition 4. *Let $f_\gamma(x) = -\ln x$. We suppose that there exists the solution $(\beta^*, Y^*) \in \bar{\mathcal{K}}$ to the equation* (18) *with*

$$Y^*(x) = \frac{1}{1-\beta^* h(x)} \text{ such that } Y^*(x) > 0 \quad \nu - a.s. \tag{19}$$

Then, there exists f-divergence minimal equivalent martingale measure Q^* and the corresponding information process is of the form:

$$\mathbf{I}(P_T|Q_T^*) = T\left\{ \frac{1}{2}\left(\beta^*\sigma\right)^2 + \int_{\mathbb{R}} \left(\ln\left(1-\beta^* h(x)\right) + \frac{1}{1-\beta^* h(x)} - 1 \right) \nu(dx) \right\}. \tag{20}$$

The integral in the right-hand side is well-defined.

Proof: We start from the proof of the following equality:

$$E_P\left[-\ln Z_T\right] = T\left\{\frac{1}{2}(\beta\sigma)^2 - \int_{\mathbb{R}} \left(\ln Y(x) - Y(x) + 1\right)\nu(dx)\right\}. \tag{21}$$

The density process Z is a positive local martingale. For $\epsilon > 0$ we put $\tau_\epsilon = \inf\{0 \le t \le T | Z_t \le \epsilon\}$ with $\inf\{\emptyset\} = +\infty$. We can write for $\forall t \in [0, T]$:

$$\{\tau_\epsilon < t\} = \bigcup_{s \in \mathbb{Q}_+, \quad s<t} \left\{ Z_s \in (0, \epsilon] \right\}$$

and since Z is adapted process, the right-hand side above is $\mathcal{F}_t$ measurable. Hence, τ_ϵ is a stopping time. Next, we show that $\tau_\epsilon \to \infty$ as $\epsilon \to 0$.

Let us take the sequence $(\epsilon_n)_{n\in\mathbb{N}}$. Since $\{\tau_{\epsilon_n} \le T\}$ is $\mathcal{F}_T$-measurable, we can write

$$Q(\tau_{\epsilon_n} \le T) = E_P\left[Z_{\tau_{\epsilon_n}} 1_{\{\tau_{\epsilon_n} \le T\}}\right] \le \epsilon_n P(\tau_{\epsilon_n} \le T).$$

Therefore,

$$Q\left(\bigcap_{n\in\mathbb{N}} \{\tau_{\epsilon_n} \le T\}\right) \le \lim_{\epsilon_n \to 0} \epsilon_n P(\tau_{\epsilon_n} \le T) = 0.$$

Using the fact that the measure Q is locally absolutely continuous with respect to measure P we can define the P-null set as

$$\mathcal{N} = \left\{\omega \in \Omega : \omega \in \bigcap_{n\in\mathbb{N}} \{\tau_{\epsilon_n} \le T\}\right\}.$$

Then, the complement of $\mathcal{N}$ is of the form

$$\mathcal{N}^c = \left\{\omega \in \Omega : \omega \in \bigcup_{n\in\mathbb{N}} \{\tau_{\epsilon_n} > T\}\right\}.$$

Then, for $\forall\omega \in \mathcal{N}^c$ there exists ϵ_0 such that $\tau_{\epsilon_0} = \infty$. Then, for $\forall\epsilon_n < \epsilon_0$, it follows that $\tau_{\epsilon_n} \ge \tau_{\epsilon_0}$, that proves that a sequence of stopping times (τ_{ϵ_n}) is increasing to infinity as $n \to \infty$. Hence, (τ_ϵ) is a localising sequence. Then, by Ito formula we have:

$$\begin{aligned} \ln Z_{T\wedge\tau_\epsilon} = \ln Z_0 &+ \int_0^{T\wedge\tau_\epsilon} \frac{1}{Z_{s-}} dZ_s - \frac{1}{2}\int_0^{T\wedge\tau_\epsilon} \frac{1}{(Z_{s-})^2} d < Z^c >_s \\ &+ \sum_{0<s\le T\wedge\tau_\epsilon} \left(\ln Z_s - \ln Z_{s-} - \frac{1}{Z_{s-}}\Delta Z_s\right), \end{aligned} \tag{22}$$

where $\Delta Z_s = Z_s - Z_{s-}$ and $< Z^c >_s$ is a predictable variation of the continuous martingale part of Z. We remark that $\left(\int_0^{T\wedge\tau_\epsilon} \frac{1}{Z_{s-}} dZ_s\right)_{t\in[0,T]}$ is a $(P,\mathbb{F})$-martingale started from zero, since it is a stochastic integral with respect to $(P,\mathbb{F})$-martingale Z (see [23] Chapter I.4d) and since $Z_{s-} \geq \epsilon > 0$ on the stochastic interval $[0, T\wedge\tau_\epsilon]$.

Let us introduce $F(x,s) = \left(\ln\left(1+\frac{x}{Z_{s-}}\right) - 1 - \frac{x}{Z_{s-}}\right)$, then by Theorem 1.8 in [23] we get

$$E_P \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} F(x,s)\mu_Z(ds,dx) = E_P \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} F(x,s)\nu_Z(ds,dx),$$

where μ_Z and ν_Z are measure of jumps and its compensator of Z.

Finally, taking the expectation in (22) and using the fact that $Z_0 = 1$, we have:

$$\begin{aligned} E_P\left[-\ln Z_{T\wedge\tau_\epsilon}\right] = E_P \Bigg[&\frac{1}{2}\int_0^{T\wedge\tau_\epsilon} \frac{1}{(Z_{s-})^2} d < Z^c >_s \\ &- \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} \left(\ln\left(1+\frac{x}{Z_{s-}}\right) - 1 - \frac{x}{Z_{s-}}\right)\nu_Z(ds,dx)\Bigg]. \end{aligned} \tag{23}$$

Since $Z = \mathcal{E}(M)$, where M is defined in (17), then $dZ_t = Z_{t-}dM_t$, and in particular $dZ_t^c = Z_t dM_t^c$ and $\Delta Z_t = Z_{t-}\Delta M_t$. In addition, (17) implies that for $\forall t \in [0,T]$, $M_t^c = \int_0^t \beta\sigma dW_s$ and $\Delta M_t = (Y(\Delta L_t) - 1)$. Hence, $< Z^c >_t = (Z_{t-})^2(\beta\sigma)^2 dt$ and $\Delta Z_t = Z_{t-}\left(Y(\Delta L_t) - 1\right)$. Then,

$$E_P\left[-\ln Z_{T\wedge\tau_\epsilon}\right] = E_P\left[T\wedge\tau_\epsilon\right]\left(\frac{1}{2}(\beta\sigma)^2 - \int_{\mathbb{R}} \left(\ln Y(x) - Y(x) + 1\right)\nu(dx)\right). \tag{24}$$

Since $\frac{1}{2}(\beta\sigma)^2 - \int_{\mathbb{R}}\left(\ln Y(x) - Y(x) + 1\right)\nu(dx) \geq 0$ and $T\wedge\tau_\epsilon \leq T$, using the Lebesgue dominated convergence theorem, we can pass to the limit on the right hand side of (23).

It remains to prove

$$\lim_{\epsilon\to 0} E_P \ln Z_{T\wedge\tau_\epsilon} = E_P \ln Z_T.$$

We can write

$$E_P \ln Z_{T\wedge\tau_\epsilon} = E_P \ln Z_T + E_P\left[\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon < T\}}\right] - E_P\left[\ln Z_T 1_{\{\tau_\epsilon < T\}}\right]. \tag{25}$$

We show that two last terms in (25) tend to zero as $\epsilon \to 0$. Let, $\hat{Z}_t = \frac{1}{Z_t}$ for all $t \in [0,T]$ and the process $\hat{Z}$ is a Q-martingale. Then, by maximal inequalities for positive martingales

$$Q\left(\tau_\epsilon < T\right) \leq Q\left(\sup_{0\leq t\leq T} \hat{Z}_t \geq \frac{1}{\epsilon}\right) \leq E_Q \hat{Z}_T \cdot \epsilon = \epsilon.$$

Finally,

$$P\left(\tau_\epsilon < T\right) = E_Q\left(\hat{Z}_T 1_{\left\{\sup\limits_{0\leq t\leq T} \hat{Z}_t \geq \frac{1}{\epsilon}\right\}}\right) \to 0, \tag{26}$$

as $\epsilon \to 0$ since $E_Q \hat{Z}_T = 1$. Since $\ln Z_T$ is P-integrable, using (26) we obtain that

$$\lim_{\epsilon\to 0} E_P\left[\ln Z_T 1_{\{\tau_\epsilon < T\}}\right] = 0.$$

Since $Z_{\tau_\epsilon} \leq \epsilon$ for $\epsilon < 1$ we get

$$E_P\left[\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon < T\}}\right] \leq 0,$$

and from concavity $\ln x,\ x > 0$

$$E_P\left[\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon < T\}}\right] \geq E_P\left[\ln Z_T 1_{\{\tau_\epsilon < T\}}\right].$$

Hence,

$$\lim_{\epsilon\to 0} E_P\left[\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon < T\}}\right] = 0,$$

as $\epsilon \to 0$. Thus,

$$\lim_{\epsilon\to 0} E_P \ln Z_{T\wedge\tau_\epsilon} = E_P \ln Z_T.$$

We remark that from (24) it is clear that the process $\ln Z$ is integrable if and only if $\int_{\mathbb{R}} \left(\ln Y(x) - Y(x)\right) \nu(dx)$ exists. Thus, we have proved the equality (21).

From Definition 3, we have that

$$\mathbf{I}(P_T|Q_T^*) = \inf_{(\beta,Y)\in\bar{\mathcal{K}}} E_P\left[-\ln Z_T\left(\beta, Y\right)\right].$$

Next, we prove that the Girsanov's parameters $(\beta^*, Y^*) \in \bar{\mathcal{K}}$, defined by (19) is the minimal point of convex function

$$G(\beta, Y) = \frac{1}{2}(\beta\sigma)^2 - \int_{\mathbb{R}} (\ln Y(x) - Y(x) + 1)\nu(dx).$$

Since we minimise the function $G(\beta, Y)$ over all $(\beta, Y) \in \bar{\mathcal{K}}$, then such (β, Y) should satisfy the martingale condition:

$$b + \beta\sigma^2 + \int_{\mathbb{R}} (Y(x) - 1)h(x)\nu(dx) = 0, \tag{27}$$

as well as the optimal parameters (β^*, Y^*):

$$b + \beta^*\sigma^2 + \int_{\mathbb{R}} (Y^*(x) - 1)h(x)\nu(dx) = 0. \tag{28}$$

We assume that

$$1 - \frac{1}{Y^*(x)} = \beta^* h(x). \tag{29}$$

We want to prove that

$$G(\beta, Y) \geq G(\beta^*, Y^*). \tag{30}$$

We remark, that G is a convex function with respect to β and Y. Then,

$$\begin{aligned}
G(\beta, Y) - G(\beta^*, Y^*) & \\
&\geq \beta^*\sigma^2(\beta - \beta^*) + \int_{\mathbb{R}} \left(1 - \frac{1}{Y^*(x)}\right)(Y(x) - Y^*(x))\nu(dx) \\
&= \beta^*\sigma^2(\beta - \beta^*) + \int_{\mathbb{R}} \beta^* h(x)(Y(x) - Y^*(x))\nu(dx) \\
&= 0,
\end{aligned}$$

where we use the convexity of G for the inequality, the assumption (29) for the first equality and the last equality follows from (27) and (28). Thus, we have proved (30). The uniqueness of (β^*, Y^*) follows from the strong convexity of function $G(\beta, Y)$ when $\sigma^2 > 0$, and in the case $\sigma^2 = 0$, directly from martingale conditions (27) and (28).

□

Proposition 5. *Let $f_\gamma(x) = x \ln x$. We suppose that there exists the solution $(\beta^*, Y^*) \in \bar{\mathcal{K}}$ to the equation (18) with*

$$Y^*(x) = e^{\beta^* h(x)}. \tag{31}$$

Then there exists f-divergence minimal equivalent martingale measure Q^* and the corresponding information process is of the form:

$$\mathbf{I}(Q_T^*|P_T) = T\left\{\frac{1}{2}\,(\beta^*\sigma)^2 + \int_{\mathbb{R}} \left[e^{\beta^* h(x)}(\beta^* h(x) - 1) + 1\right] \nu(dx)\right\}. \tag{32}$$

The integral in the right hand side is well defined.

Proof:

We continue in the framework of Proposition 4 to prove the equality (32). We start from the proof of an equality

$$E_P\left[Z_T \ln Z_T\right] = T\left\{\frac{1}{2}(\beta\sigma)^2 ds + \int_{\mathbb{R}} \left(Y(x)\ln Y(x) - Y(x) + 1\right)\nu(dx)\right\}. \tag{33}$$

The density process Z is a positive local martingale. For $\epsilon > 0$ we put $\tau_\epsilon = \inf\{0 \le t \le T | Z_t \le \epsilon \text{ or } Z_t \ge \frac{1}{\epsilon}\}$ with $\inf\{\emptyset\} = +\infty$. The sequence of stopping times (τ_ϵ) is increasing to infinity as $\epsilon \to 0$. Hence, (τ_ϵ) is a localising sequence. Then, by Ito formula we have:

$$Z_{T\wedge\tau_\epsilon} \ln Z_{T\wedge\tau_\epsilon} = Z_0 \ln Z_0 + \int_0^{T\wedge\tau_\epsilon} \left(\ln Z_{s-} + 1\right) dZ_s + \frac{1}{2}\int_0^{T\wedge\tau_\epsilon} \frac{1}{(Z_{s-})^2} d < Z^c >_s$$
$$+ \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} \left((Z_{s-} + x)\ln\left(Z_{s-} + x\right) - Z_{s-}\ln Z_{s-} - (\ln Z_{s-} + 1)x\right)\mu_Z(ds, dx).$$

We remark that $\left(\int_0^{T\wedge\tau_\epsilon} \left(\ln Z_{s-} + 1\right) dZ_s\right)_{t\in[0,T]}$ is a $(P, \mathbb{F})$-martingale started from zero, since it is stochastic integral with respect to $(P, \mathbb{F})$-martingale Z (see [23] Chapter I.4d) and since $Z_{s-} \ge \epsilon$ or $Z_{s-} \le \frac{1}{\epsilon}$ on the stochastic interval $[0, T \wedge \tau_\epsilon]$.

Let us introduce the function

$$F(x, s) = \left((Z_{s-} + x)\ln\left(Z_{s-} + x\right) - Z_{s-}\ln Z_{s-} - (\ln Z_{s-} + 1)x\right).$$

Then, by Theorem 2.1.8 in [23] we get

$$E_P \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} F(x, s)\mu_Z(ds, dx) = E_P \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} F(x, s)\nu_Z(ds, dx), \tag{34}$$

where μ_Z and ν_Z are measure of jumps of process Z and its compensator .

Finally, taking the expectation of $Z_{T\wedge\tau_\epsilon}\ln Z_{T\wedge\tau_\epsilon}$ and using the fact that $Z_0 = 1$, we have:

$$E_P\left[Z_{T\wedge\tau_\epsilon}\ln Z_{T\wedge\tau_\epsilon}\right] = E_P\left[\frac{1}{2}\int_0^{T\wedge\tau_\epsilon}\frac{1}{(Z_{s-})}d < Z^c >_s\right.$$
$$\left.+\int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}}\left((Z_{s-}+x)\ln\left(Z_{s-}+x\right) - Z_{s-}\ln Z_{s-} - (\ln Z_{s-}+1)x\right)\nu_Z(ds,dx)\right].$$

Since $Z = \mathcal{E}(M)$, where M is defined in (17), then $< Z^c >_t = Z_{t-}(\beta\sigma)^2 dt$ and $\Delta Z_t = Z_{t-}\left(Y(\Delta L_t) - 1\right)$. Then,

$$E_P\left[Z_{T\wedge\tau_\epsilon}\ln Z_{T\wedge\tau_\epsilon}\right] = E_P\left[\frac{1}{2}\int_0^{T\wedge\tau_\epsilon} Z_{s-}(\beta\sigma)^2 ds\right.$$
$$\left.+\int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}} Z_{s-}\left(Y(x)\ln Y(x) - Y(x) + 1\right)\nu(dx)\right]. \tag{35}$$

Since $\tau_\epsilon \to \infty$ as $\epsilon \to 0$ and $x\ln x - x + 1 \geq 0$ for all $x > 0$, by Lebesgue's monotone convergence theorem we can pass to the limit on the right-hand side of (35).

It remains to prove

$$\lim_{\epsilon\to 0} E_P\left[Z_{T\wedge\tau_\epsilon}\ln Z_{T\wedge\tau_\epsilon}\right] = E_P\left[Z_T\ln Z_T\right].$$

We can write

$$\begin{aligned} E_P\left[Z_{T\wedge\tau_\epsilon}\ln Z_{T\wedge\tau_\epsilon}\right] = E_P\left[Z_T\ln Z_T\right] \quad &+ \quad E_P\left[Z_{\tau_\epsilon}\ln Z_{\tau_\epsilon}1_{\{\tau_\epsilon<T\}}\right] \\ &- \quad E_P\left[Z_T\ln Z_T 1_{\{\tau_\epsilon<T\}}\right]. \end{aligned} \tag{36}$$

We show that two last terms in (36) tend to zero as $\epsilon \to 0$.

Since $Z_T\ln Z_T$ is P-integrable and $\lim_{\epsilon\to 0} P\left(\tau_\epsilon < T\right) = 0$, we get that

$$\lim_{\epsilon\to 0} E_P\left[Z_T\ln Z_T 1_{\{\tau_\epsilon<T\}}\right] = 0.$$

Using the inequality $x\ln x \geq \frac{1}{e}$ for all $x \in \mathbb{R}_+$, we have for $0 \leq \epsilon \leq \frac{1}{e}$

$$-\frac{1}{e}\cdot P\left(\tau_\epsilon < T\right) \leq E_P\left[Z_{\tau_\epsilon}\ln Z_{\tau_\epsilon}1_{\{\tau_\epsilon<T\}}\right] \leq 0.$$

Hence,

$$\lim_{\epsilon\to 0} E_P\left[Z_{\tau_\epsilon}\ln Z_{\tau_\epsilon} 1_{\{\tau_\epsilon<T\}}\right]=0$$

and the equality (33) holds.

From Definition 3 we have that

$$\mathbf{I}(Q_T^*|P_T)=\inf_{(\beta,Y)\in\bar{\mathcal{K}}} E_P\left[Z_T(\beta,Y)\ln Z_T(\beta,Y)\right].$$

Next, we prove that the Girsanov's parameters $(\beta^*,Y^*)\in\bar{\mathcal{K}}$, defined by (31) is the minimal point of convex function

$$G(\beta,Y)=\frac{1}{2}(\beta\sigma)^2+\int_{\mathbb{R}}(Y(x)\ln Y(x)-Y(x)+1)\nu(dx).$$

Since we minimise the function $G(\beta,Y)$ over all $(\beta,Y)\in\bar{\mathcal{K}}$, then such (β,Y) should satisfy the martingale conditions (27) and (28).

We assume that

$$Y^*(x)=e^{\beta^* h(x)}. \tag{37}$$

We remark, that G is a convex function with respect to β and Y. Then,

$$\begin{aligned}
&G(\beta,Y)-G(\beta^*,Y^*)\\
&\quad\geq \beta^*\sigma^2(\beta-\beta^*)+\int_{\mathbb{R}}\left(\ln Y^*(x)(Y(x)-Y^*(x))\right)\nu(dx)\\
&\quad= \beta^*\sigma^2(\beta-\beta^*)+\int_{\mathbb{R}}\left(\beta^* h(x)(Y(x)-Y^*(x))\right)\nu(dx)\\
&\quad= 0,
\end{aligned}$$

where we use the convexity of G for the inequality, the assumption (37) for the first equality and the last equality follows from (27) and (28). Thus, we have proved that $G(\beta,Y)\geq G(\beta^*,Y^*)$. The uniqueness of (β^*,Y^*) follows from the strong convexity of function $G(\beta,Y)$ when $\sigma^2>0$, and in the case $\sigma^2=0$, directly from martingale conditions (27) and (28).

□

Proposition 6. *Let $f(x)=-\frac{1}{q}x^q,\quad q\in(-\infty,0)\cup(0,1)$. We suppose that there exists the solution $(\beta^*,Y^*)\in\bar{\mathcal{K}}$ to the equation (18) with*

$$Y^*(x)=(1+(q-1)\beta^* h(x))^{\frac{1}{q-1}}\ \text{ such that } 1+(q-1)\beta^* h(x)>0\ \nu\text{-}a.s. \tag{38}$$

Then, there exists f-divergence minimal equivalent martingale measure Q^* and the corresponding Hellinger type process is of the form:

$$H_T^{(q*)} = 1 + T\left\{\frac{1}{2}q(q-1)\,(\beta^*\sigma)^2 + \int_{\mathbb{R}}\Big((1+(q-1)\beta^* h(x))^{\frac{q}{q-1}} + q\,(1+(q-1)\beta^* h(x))^{\frac{1}{q-1}} - 1\Big)\,\nu(dx)\right\}. \tag{39}$$

The integral in the right-hand side is well-defined.

Proof: We continue in the framework of Proposition 4 to prove the equality (39) and we start from the proof of an equality

$$E_P Z_T^q = 1+T\left\{\frac{1}{2}q(q-1)(\beta\sigma)^2+\int_{\mathbb{R}}\left(Y^q(x) - q(Y(x)-1) - 1\right)\nu(dx)\right\}. \tag{40}$$

The density process Z is a positive local martingale. For $\epsilon > 0$ we put $\tau_\epsilon = \inf\{0 \le t \le T | Z_t \le \epsilon\}$ with $\inf\{\emptyset\} = +\infty$. The sequence of stopping times (τ_ϵ) is increasing to infinity as $\epsilon \to 0$. Hence, (τ_ϵ) is a localising sequence. Then, by Ito formula we have:

$$Z^q_{T\wedge\tau_\epsilon} = 1 + q\int_0^{T\wedge\tau_\epsilon} Z_{s-}^{q-1} dZ_s + \frac{1}{2}q(q-1)\int_0^{T\wedge\tau_\epsilon}(Z_{s-})^{q-2} d < Z^c >_s + \int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}}\left[Z_{s-}\left(\left(1+\frac{x}{Z_{s-}}\right)^q - 1\right)\right]\mu_Z(ds,dx). \tag{41}$$

We remark that $\left(\int_0^{T\wedge\tau_\epsilon}(Z_{s-})^{q-1}\,dZ_s\right)_{t\in[0,T]}$ is a $(P,\mathbb{F})$-martingale started from zero. Let us introduce the function

$$F(x,s) = \left((Z_{s-}+x)\ln(Z_{s-}+x) - Z_{s-}\ln Z_{s-} - (\ln Z_{s-}+1)x\right).$$

Then, by the projection theorem (Theorem 2.1.8 in [23]) we get

$$E_P\int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}} F(x,s)\mu_Z(ds,dx) = E_P\int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}} F(x,s)\nu_Z(ds,dx),$$

where μ_Z and ν_Z are measure of jumps of process Z and its compensator .

Finally, taking the expectation in (41) we have:

$$E_P Z^q_{T\wedge\tau_\epsilon} = 1 + E_P\left[\frac{1}{2}q(q-1)\int_0^{T\wedge\tau_\epsilon}(Z_{s-})^{q-2} d < Z^c >_s + \int_0^{T\wedge\tau_\epsilon}\int_{\mathbb{R}}\left[Z_{s-}\left(\left(1+\frac{x}{Z_{s-}}\right)^q - 1\right)\right]\nu_Z(ds,dx)\right].$$

Using the relation between $Z = \mathcal{E}(M)$, where M is defined in (17) and the Levy process L, we get

$$
\begin{aligned}
(42)\quad E_P Z^q_{T\wedge\tau_\epsilon} = 1 + E_P \Bigg[\frac{1}{2} q(q-1) \int_0^{T\wedge\tau_\epsilon} Z^q_{s-} (\beta\sigma)^2 ds \\
+ \int_0^{T\wedge\tau_\epsilon} \int_{\mathbb{R}} Z^q_{s-} \left[Y^q(x) - q(Y(x)-1) - 1 \right] \nu_z(ds,dx) \Bigg].
\end{aligned}
$$

We remark that $\tau_\epsilon \to \infty$ as $\epsilon \to 0$. Since for $q \in (0,1)$, $q(q-1) < 0$ and $x^q - qx - 1 \le 0$ and for $q \in (-\infty, 0)$, $q(q-1) > 0$ and $x^q - qx - 1 \ge 0$, we conclude that the right hand side of the above expression contains the integral of some negative function. Then, by Lebesgue's monotone convergence theorem we can pass to the limit on the right-hand side of (42). It remains to show that the left-hand side of (42) converges to $\mathbb{E} Z^q_T$.

We can write

$$
(43)\qquad E_P Z^q_{T\wedge\tau_\epsilon} = E_P Z^q_T + E_P \left[Z^q_{\tau_\epsilon} 1_{\{\tau_\epsilon < T\}} \right] - E_P \left[Z^q_T 1_{\{\tau_\epsilon < T\}} \right].
$$

We show that two last terms in (43) tend to zero as $\epsilon \to 0$.

Since Z^q_T is P-integrable and $\lim_{\epsilon\to 0} P(\tau_\epsilon < T) = 0$, we get that

$$
\lim_{\epsilon\to 0} E_P \left[Z^q_T 1_{\{\tau_\epsilon < T\}} \right] = 0.
$$

For the second term we distinguish two cases:

$$
q \in (-\infty, 0) \text{ and } q \in (0,1).
$$

For $q \in (0,1)$ we have:

$$
E_P \left[Z^q_{\tau_\epsilon} 1_{\{\tau_\epsilon < T\}} \right] \le \epsilon^q \cdot P(\tau_\epsilon < T) \to 0
$$

as $\epsilon \to 0$. In the case $q \in (-\infty, 0)$ we have:

$$
E_P \left[Z^q_{\tau_\epsilon} 1_{\{\tau_\epsilon < T\}} \right] \le E_Q \left[\hat{Z}^{1-q}_{\tau_\epsilon} 1_{\{\tau_\epsilon < T\}} \right],
$$

where $\hat{Z}_{\tau_\epsilon} = \frac{1}{Z_{\tau_\epsilon}}$. From maximal inequalities for the martingales we have:

$$
E_Q \left[\sup_{0\le t\le T} \hat{Z}_t \right]^{1-q} \le c(q) E_Q \left[\hat{Z}^{1-q}_T \right] = c(q) E_P \left[Z^q_T \right] < \infty,
$$

where $c(q)$ is a constant. In addition, $\lim_{\epsilon\to 0} Q(\tau_\epsilon < T) = 0$, then

$$\lim_{\epsilon\to 0} E_Q\left[\hat{Z}_{\tau_\epsilon}^{1-q} 1_{\{\tau_\epsilon<T\}}\right] = 0.$$

And we have proved, that

$$\lim_{\epsilon\to 0} E_P Z^q_{T\wedge\tau_\epsilon} = E_P Z^q_T.$$

Next, we prove that the Girsanov's parameters $(\beta^*, Y^*) \in \bar{\mathcal{K}}$, defined by (38) is the minimal point of convex function

$$G(\beta, Y) = \frac{1}{2}q(q-1)(\beta\sigma)^2 + \int_{\mathbb{R}} (Y^q(x) - q(Y(x)-1) - 1)\nu(dx),\ q \in (-\infty, 0),$$

and the minimal point of the convex function $-G(\beta, Y)$ if $q \in (0,1)$. In this proof we consider the case when $q \in (-\infty, 0)$ and we minimise the convex function $G(\beta, Y)$ over all $(\beta, Y) \in \bar{\mathcal{K}}$. Then such (β, Y) should satisfy the martingale conditions (27) and (28).

We assume that

$$Y^{*q-1}(x) - 1 = (q-1)\beta^* h(x). \tag{44}$$

Since G is a convex function with respect to β and Y, then,

$$\begin{aligned}
G(\beta, Y) \quad - \quad & G(\beta^*, Y^*) \\
\geq \quad & q(q-1)\beta^*\sigma^2(\beta-\beta^*) + \int_{\mathbb{R}} \left(q\left(Y^{q-1}(x)-1\right)\right)(Y(x)-Y^*(x))\nu(dx) \\
= \quad & q(q-1)\beta^*\sigma^2(\beta-\beta^*) + \int_{\mathbb{R}} q(q-1)\beta^* h(x)(Y(x)-Y^*(x))\nu(dx) \\
= \quad & 0,
\end{aligned}$$

where we use the convexity of G for the inequality, the assumption (44) for the first equality and the last equality follows from (27) and (28). Thus, we have proved that $G(\beta, Y) \geq G(\beta^*, Y^*)$. The uniqueness of (β^*, Y^*) follows from the strong convexity of the functions $G(\beta, Y)$, $q \in (-\infty, 0)$ and $-G(\beta, Y)$, $q \in (0,1)$ when $\sigma^2 > 0$, and in the case $\sigma^2 = 0$, directly from martingale conditions (27) and (28).

□

3.2. **f-divergence minimal martingale measures on initially enlarged filtration.** In this section we consider the standard f-divergences and we prove that the density process $Z^*(\xi)$ of the f-minimal equivalent martingale measure on the initially enlarged filtration coincides with the density process Z^* of the f-minimal equivalent martingale measure on the reference filtration $\mathbb{F}$ up to the random factor which is $\sigma(\xi)$-measurable random variable $Z_0(\xi)$. The next auxiliary lemma establishes that the minimal martingale measure for the conditional dual problem to the utility maximisation problem on the enlarged filtration, coincides with Q^*.

Lemma 2. *Let Z^* be the f-divergence minimal equivalent martingale measure for the geometric Levy process $S = \mathcal{E}(L)$ defined in Definition 3. Then*

$$\inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{G})} \mathbb{E}\left[f(\tilde{Z}(\xi))\right] = \inf_{Z_0} \mathbb{E}\left[f(Z_0(\xi)Z_T^*)\right]. \tag{45}$$

Proof: Since the process $S\tilde{Z}_T(u)$ is $(P,\mathbb{F})$-martingale, then the measure Q^u given by the density process $\tilde{Z}(u)$ belongs to the set $\mathcal{M}(\mathbb{F})$. Then the following inequality holds:

$$E_P\left[f\left(\tilde{Z}_T(u)\right)\right] \geq E_P\left[f\left(Z_T^*\right)\right]. \tag{46}$$

Using the fact that we consider the f-divergences which are invariant under scaling and (46) we get that

$$E_P\left[f\left(Z_0(u)\tilde{Z}_T(u)\right)\right] \geq E_P\left[f\left(Z_0(u)Z_T^*\right)\right].$$

Integrating with respect to μ we get that

$$\inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{G})} \mathbb{E}\left[f\left(Z_0(\xi)\tilde{Z}_T(\xi)\right)\right] \geq \inf_{Z_0} \mathbb{E}\left[f\left(Z_0(\xi)Z_T^*\right)\right].$$

From the other side, if a measure $Q \in \mathcal{M}(\mathbb{F})$ then the price process S is a $(Q,\mathbb{F})$-martingale, then by immersion property (see [24]) S is a $(\mathbb{Q},\mathbb{G})$-martingale, where $\mathbb{Q}(A\times B) = Q(A)$ for $A \in \mathcal{F}$ and $B \in \sigma(\xi)$. Then $\mathbb{Q} \in \mathcal{M}(\mathbb{G})$. And we can conclude that

$$\inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{G})} \mathbb{E}\left[f\left(Z_0(\xi)\tilde{Z}_T(\xi)\right)\right] \leq \inf_{Z_0} \mathbb{E}\left[f\left(Z_0(\xi)Z_T^*\right)\right].$$

Hence, the relation (45) has been proved.

□

It follows from the definition of the density process of $\mathbb{Q} \in \mathcal{M}(\mathbb{G})$ that $Z(\xi) = Z_0(\xi)\tilde{Z}(\xi)$, where the random variable $Z_0(\xi)$ is such that $E_\mu Z_0(\xi) =$

1. By the Definition 1, Proposition 4 and Lemma 2 we should minimise the function

$$F(\lambda, Z_0, \tilde{Z}_T) = \int_{\mathbb{R}_+} E_P\left[f\left(\lambda Z_0(u)\tilde{Z}_T(u)\right) + \lambda Z_0(u)(x + g(u))\right] d\mu(u).$$

In next proposition we find the solution to the minimisation problem F in the cases of the special f-divergences.

Proposition 7. *Let Z_T^* be a density process of the minimal f-divergence equivalent martingale measure $Q^* \in \mathcal{M}(\mathbb{F})$. We assume that f is a strictly convex function and for all $\lambda > 0$ and $t \in [0, T]$ we have that function $Z_t^* f'(\lambda Z_t^*)$ is integrable $\mathbb{P}-a.s.$ Then, if the f-minimal equivalent measure $\mathbb{Q}^*$ exists, it is unique and it has the following structure:*

$$\frac{d\mathbb{Q}^*|_{\mathcal{G}_T}}{d\mathbb{P}|_{\mathcal{G}_T}} = Z_0(\xi) Z_T^*, \tag{47}$$

where $Z_0(\xi)$ is a $\mathcal{G}$-measurable random variable, such that

$$E_\mu Z_0(\xi) = 1 \tag{48}$$

Let us denote by $\tilde{\lambda}(u) := \lambda Z_0(u)$. Then for any random initial capital $x + g(\xi) \in (\underline{x}, \infty)$ there exists unique $\tilde{\lambda}^*(u) > 0$ such that for any $u \in supp(\mu)$ defined uniquely from the equation

$$E_P\left[-Z_T^* f'\left(\tilde{\lambda}^*(u) Z_T^*\right)\right] = x + g(u). \tag{49}$$

Moreover, for any $u \in supp(\mu)$, $Z_0^*(u)$ verify:

$$Z_0^*(u) = \frac{\tilde{\lambda}^*(u)}{E_\mu \tilde{\lambda}^*(\xi)}. \tag{50}$$

In particular, if $f'(x) = \frac{1}{\gamma}(\ln x - \ln\gamma)$, for $\gamma > 0$ then

$$Z_0^*(u) = \frac{\exp\left(-\gamma g(u)\right)}{E_\mu\left[\exp\left(-\gamma g(\xi)\right)\right]}, \tag{51}$$

and for $f'(x) = -x^{\frac{1}{p-1}}$, for $p < 1$ we get

$$Z_0^*(u) = \frac{(x + g(u))^{p-1}}{E_\mu\left[(x + g(\xi))^{p-1}\right]}. \tag{52}$$

Proof:

We want to minimise function $F(\lambda, Z_0(u))$ subject to (48). We consider the unconstrained optimisation problem such that to minimise function

$$F(\tilde{\lambda}) = \int_{\mathbb{R}} E_P\left[f\left(\tilde{\lambda}(u)Z_T^*\right) + \tilde{\lambda}(u)(x+g(u))\right] d\mu(u) \tag{53}$$

over all measurable functions $\tilde{\lambda} : \mathbb{R}_+ \to \mathbb{R}_+$.

Let us assume that the minimiser λ^* of function $F(\tilde{\lambda})$ satisfies to the following equation:

$$E_P\left[-Z_T^* f'\left(\tilde{\lambda}(u)Z_T^*\right)\right] = x + g(u), \quad \mu - a.s.. \tag{54}$$

Since $E_P\left[f'\left(\tilde{\lambda}(u)Z_T^*\right)\right] < \infty$ for all $\tilde{\lambda}(u) > 0$, one can conclude that function $E_P\left[-f'\left(\tilde{\lambda}(u)Z_T^*\right)\right]$ is a continuous, monotonically decreasing function of $\tilde{\lambda}(u)$ with values in $(\underline{x}, \infty)$. This guaranties the existence of $\tilde{\lambda}^*$.

We set

$$F(\tilde{\lambda}) = \int_{\mathbb{R}} E_P\left[f\left(\tilde{\lambda}(u)Z_T^*\right) + \tilde{\lambda}(u)(x+g(u))\right] d\mu(u),$$

$$F(\tilde{\lambda}^*) = \int_{\mathbb{R}} E_P\left[f\left(\tilde{\lambda}^*(u)Z_T^*\right) + \tilde{\lambda}^*(u)(x+g(u))\right] d\mu(u).$$

We have to prove for $\forall\tilde{\lambda} > 0$ that

$$F(\tilde{\lambda}) \geq F(\tilde{\lambda}^*). \tag{55}$$

We remark that we consider the standard f-divergences, which are strictly convex with respect to $\tilde{\lambda}$, then F is a convex function. So, we obtain

$$\begin{aligned}
&F(\tilde{\lambda}) - F(\tilde{\lambda}^*) \\
&\qquad \geq \int_{\mathbb{R}} \left(E_P\left[f'\left(\tilde{\lambda}^*(u)Z_T^*\right) Z_T^*\right] + x + g(u)\right) (\lambda(u) - \lambda^*(u))\, d\mu(u) \\
&\qquad = 0,
\end{aligned}$$

where we use the convexity of $F(\tilde{\lambda})$ for the inequality and the assumption that $x + g(u) = E_P\left[-Z_T^* f'\left(\tilde{\lambda}^*(u)Z_T^*\right)\right]$ for the equality. Since (55) is proved, $\tilde{\lambda}^*$ defined from (54) is a minimiser of (53). Moreover, under the condition (14) on the initial capital x, the standard f-divergence functions are strongly convex. Therefore, the uniqueness of $\tilde{\lambda}^*$ follows from the strong convexity of $F(\lambda)$.

Now, $\lambda^* Z_0^*(u) = \tilde{\lambda}^*(u)$ and since $E_\mu Z_0^*(\xi) = 1$, the equality (50) follows.

In particular, in the case of $f^{'}(x) = \frac{1}{\gamma}(\ln x - \ln \gamma)$ we get

$$\tilde{\lambda}^*(u) = \gamma \exp\left(-\gamma(x + g(u)) - E_P\left[Z_T^* \ln Z_T^*\right]\right), \tag{56}$$

and then (51). And for $f^{'}(x) = -x^{\frac{1}{p-1}}$, $p < 1$

$$\tilde{\lambda}^*(u) = \frac{(x + g(u))^{p-1}}{\left(E_P\left[Z_T^*\right]^{\frac{p}{p-1}}\right)^{p-1}}, \tag{57}$$

and then (52) follows.

□

4. Solution to utility maximisation problem and indifference prices formulas

We consider the filtered probability space $(\Omega \times \mathbb{R}, \mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+), \mathbb{G}, \mathbb{P})$, where $\mathcal{G}_t = \bigcap_{s>t}(\mathcal{F}_s \otimes \sigma(\xi))$. (For more accurate description of the enlarged probability space see Section 2.) Let x be an initial endowment, $G_T = g(\xi)$ be a payoff function of the European type option, g is a positive Borel function and $\Pi(\mathbb{G})$ be the set of the admissible and self-financing portfolios.

Proposition 8. *Let $V(x, g)$ is defined by* (11) *and let the conditions of Proposition 7 are satisfied.*

(i) Then,

$$V(x, g) = \int_{\mathbb{R}} E_P\left[U(-f^{'}(\tilde{\lambda}^*(u) Z_T^*)\right] d\mu(u), \tag{58}$$

and for any initial capital $x + g(u) \in (\bar{x}, \infty)$, $\tilde{\lambda}^*(u)$ is unique solution of the equation (49).

(ii) In particular, in the case of exponential utility $U(x) = 1 - e^{-\gamma x}$, we get

$$V(x,g) = 1 - \exp\Bigg\{ -\gamma x - T\Big\{\frac{1}{2}\left(\beta^*\sigma\right)^2 + \int_{\mathbb{R}} \left[e^{\beta^* h(x)}(\beta^* h(x) - 1) + 1\right]\nu(dx)\Big\}\Bigg\} \times \int_{\mathbb{R}} \exp\{-\gamma g(u)\} d\mu(u), \tag{59}$$

where (β^* *is solution to the equation* (18) *with* $Y^*(x) = e^{\beta^* h(x)}$.

For logarithmic utility $U(x) = \ln x$,

$$V(x,g) = \int_{\mathbb{R}} \ln(x + g(u)) d\mu(u) + T\Big\{\frac{1}{2}\left(\beta^*\sigma\right)^2 + \int_{\mathbb{R}} \left(\ln\left(1 - \beta^* h(x)\right) + \frac{1}{1 - \beta^* h(x)} - 1\right)\nu(dx)\Big\}, \tag{60}$$

where β^* *is solution to the equation* (18) *with* $Y^*(x) = \frac{1}{1-\beta^* h(x)}$.

And for power utility $U(x) = \frac{x^p}{p}$, $p \in (-\infty, 0) \cup (0,1)$,

$$V(x,g) = \frac{1}{p}\int_{\mathbb{R}} (x + g(u))^p \Bigg(1 + T\Big\{\frac{1}{2}q(q-1)\left(\beta^*\sigma\right)^2 + \int_{\mathbb{R}} \left(\left(1 + (q-1)\beta^* h(x)\right)^{\frac{q}{q-1}} + q\left(1 + (q-1)\beta^* h(x)\right)^{\frac{1}{q-1}} - 1\right)\nu(dx)\Big\}\Bigg)^{1-p} d\mu(u), \tag{61}$$

where $q = \frac{p}{p-1}$ *and* β^* *is a solution to the equation* (18) *with* $Y^*(x) = (1 + (q-1)\beta^* h(x))^{\frac{1}{q-1}}$ *such that* $1 + (q-1)\beta^* h(x) > 0$ $\nu-a.s.$

Proof:

According to Proposition 3 the utility optimisation problem on initially enlarged filtration can be written in a following dual form:

$$V(x,g) = \inf_{\lambda > 0} \inf_{\mathbb{Q} \in \tilde{\mathcal{K}}} \left\{ \mathbb{E}\left[f\left(\lambda \frac{d\mathbb{Q}|_T}{d\mathbb{P}|_T}\right) + \lambda Z_0(\xi)(x + g(\xi))\right]\right\}. \tag{62}$$

From Proposition 7 we know that

$$\inf_{\mathbb{Q} \in \tilde{\mathcal{K}}} \mathbb{E}\left[f\left(\lambda \frac{d\mathbb{Q}|_T}{d\mathbb{P}|_T}\right)\right] = \mathbb{E}\left[f\left(\lambda Z_0(\xi) Z_T^*\right)\right],$$

where $\lambda Z_0(\xi) = \tilde{\lambda}(\xi)$. Since we consider the standard f-divergences which are invariant under scaling, we obtain the value function $V(x,g)$ on the initially enlarged filtration in the following dual form:

$$V(x,g) = \mathbb{E}\left[f\left(\tilde{\lambda}^*(\xi)Z_T^*\right) + \tilde{\lambda}^*(\xi)(x+g(\xi))\right], \tag{63}$$

where $\tilde{\lambda}^*(u)$ is uniquely defined from (49). Then, the equality (58) follows from (ii) of Proposition 2 .

Then, using (i), in the case of exponential utility $U(x) = 1 - e^{-\gamma x}$ with $f^{'}(x) = \frac{1}{\gamma}(\ln x - \ln \gamma)$, $\gamma > 0$, we get

$$V(x,g) = 1 - \exp\{-E_P\left[Z_T^* \ln Z_T^*\right] - \gamma x\} \int_{\mathbb{R}} \exp\{-\gamma g(u)\} d\mu(u). \tag{64}$$

Using Proposition 5 and that $E_P\left[Z_T^* \ln Z_T^*\right] = \mathbf{I}(Q_T^*|P_T)$ we obtain the formulae (59) for the value function V in the case of exponential utility.

For logarithmic utility $U(x) = \ln x$ with $f^{'}(x) = -\frac{1}{x}$,

$$V(x,g) = \int_{\mathbb{R}} \ln(x+g(u))d\mu(u) - E_P \ln Z_T^*. \tag{65}$$

Then, from Proposition 4 and using that $-E_P \ln Z_T^* = \mathbf{I}(P_T|Q_T^*)$ we get (60).

And for power utility $U(x) = \frac{x^p}{p}$ with $f^{'}(x) = -x^{\frac{1}{p-1}}$, $p < 1$,

$$V(x,g) = \frac{1}{p}\int_{\mathbb{R}} (x+g(u))^p \left(\mathbb{E}\left[Z_T^*\right]^{\frac{p}{p-1}}\right)^{1-p} d\mu(u). \tag{66}$$

From Proposition 6 and $E_P\left[Z_T^*\right]^{\frac{p}{p-1}} = H_T^{(q*)}$ we get (61).

□

We remind that a buyer's indifference price p_T^b is the solution to

$$V_T(x,0) = V_T(x - p_T^b, g). \tag{67}$$

and a seller's indifference price p_T^s is defined from

$$V_T(x,0) = V_T(x + p_T^s, -g). \tag{68}$$

Next, we apply the results of Proposition 8 to calculate the indifference prices in the cases of the logarithmic, exponential and power utilities. Then, we show that the corresponding seller's indifference prices $p_T^s(g)$ satisfy the properties of the convex risk measures, such that convexity, monotonicity and translation invariance property with respect to the claims.

Proposition 9. *Let $V(x,g)$ is defined by (11) and let the conditions of Proposition 7 are satisfied. If $\ln g(\xi)$, $\ln (x-g(\xi))$ are integrable functions and $g(u)\in(0,x)$, $(\mu-a.s.)$, then the buyer's and seller's indifference prices for logarithmic utility function $U(x)=\ln x,\ x>0$ are defined respectively from*

$$\int_{\mathbb{R}} \ln\left[1-\frac{p_T^b}{x}+\frac{g(u)}{x}\right] d\mu(u)=0 \tag{69}$$

and

$$\int_{\mathbb{R}} \ln\left[1+\frac{p_T^s}{x}-\frac{g(u)}{x}\right] d\mu(u)=0. \tag{70}$$

Moreover, then there exists the unique solutions $p_T^b, p_T^s\in[0,x]$ of the equations (69) and (70).

Moreover, $p_T^b(g)$ is concave, increasing functional and $p_T^s(g)$ is convex, decreasing functional, which both satisfy the translation invariance property: $\forall\ m\in\mathbb{R}$

$$p_T^{b,s}(g+m)=p_T^{b,s}(g)-m.$$

Proof:

To solve the indifference pricing problems (67) and (68) for the logarithmic utility we use Proposition 7. The values of the left hand sides of (67) and (68) coincide and can be obtained by taking $g=0$ in (65). To calculate the right hand side of (67) we substitute x by $x-p_T^b$. The value function on the right hand side of (68) corresponds to the situation of the selling the option and, hence, g is replaced by $-g$ and x is substituted by $x+p_T^s$ in (65). Therefore, from (67) and (65) we have

$$\int_{\mathbb{R}} \left[\ln\left(x-p_T^b+g(u)\right)+\mathbf{I}\left(P_T|Q_T^*\right)-\ln x-\mathbf{I}\left(P_T|Q_T^*\right)\right] d\mu(u)=0,$$

and hence we obtain (69). Formula (70) is obtained from the relation

$$p_T^b(g)=-p_T^s(-g).$$

Let us denote by $F(y)$ the function on $[0,x]$ of the form:

$$F(y) = \int_{\mathbb{R}} \ln\left[1 - \frac{y}{x} + \frac{g(u)}{x}\right] d\mu(u), \quad y \in [0,x]\,.$$

Function $\ln\left[1 - \frac{y}{x} + \frac{g(u)}{x}\right]$ is a well-defined decreasing function of y on $[0,x]$ if $g(u) \in (0,x)$ and

$$\frac{g(u)}{x} \leq 1 - \frac{y}{x} + \frac{g(u)}{x} \leq 1 + \frac{g(u)}{x}$$

and

$$\ln\frac{g(u)}{x} \leq \ln\left[1 - \frac{y}{x} + \frac{g(u)}{x}\right] \leq \ln\left[1 + \frac{g(u)}{x}\right].$$

Then, if $\ln g(u)$, $u \in \mathbb{R}$ is integrable function with respect to μ, then function $F(y)$ is a well-defined on $[0,x]$ and by Lebesque theorem $F(y)$ is continuous function on $[0,x]$. Since $F(x) \leq 0$ and $0 \leq F(0) < \infty$, then a solution to equation $F(y) = 0$ exists by the mean-value theorem.

Next, the uniqueness of solution of $F(y) = 0$ follows from the fact that $F(y)$ is a strictly decreasing function. In fact, let us denote by $\tilde{f}(y,u)$, $(y,u) \in [0,x] \times \mathbb{R}$ the integrable with respect to μ function of the form:

$$\tilde{f}(y,u) = \ln\left[1 - \frac{y}{x} + \frac{g(u)}{x}\right].$$

The function $\tilde{f}(y,u)$ is continuous in y and u on $[0,x] \times \mathbb{R}$ and its derivative

$$\tilde{f}_y(y,u) = -\frac{1}{x - y + g(u)}$$

is also continuous on $[0,x] \times \mathbb{R}$.

Moreover, if $g(u) \in (0,x)$, then $x - y + g(u) > 0$ and $\tilde{f}_y(y,u) < 0$ for all $(y,u) \in [0,x] \times \mathbb{R}$. Additionally, for $x > 0$:

$$\tilde{f}_y(y,u) < -\frac{1}{2x} < 0.$$

Then,

$$\frac{\partial}{\partial y} F(y) = \int_{\mathbb{R}} \tilde{f}_y(y,u) d\mu(u)$$

and for $x > 0$:

$$\frac{\partial}{\partial y} F(y) \leq -\frac{1}{2x} < 0.$$

Hence, $F(y)$ is a strictly decreasing function of y on $[0, x]$. In the case of the sellers indifference price, we denote

$$F(y) = \int_{\mathbb{R}} \ln\left[1 + \frac{y}{x} - \frac{g(u)}{x}\right] d\mu(u), \quad y \in [0, x].$$

Since $\ln\left[1 + \frac{y}{x} - \frac{g(u)}{x}\right]$ is a well-defined increasing function on $[0, x]$, the condition

$$\int_{\mathbb{R}} \ln\left(x - g(u)\right) d\mu(u) < \infty$$

implies that function $F(y)$ is a well-defined continuous function on $[0, x]$. Since $F(0) \leq 0$ and $F(x) \geq 0$, a solution to equation $F(y) = 0$ exists by the mean-value theorem. The uniqueness of solution follows from the fact that $F(y)$ is a strictly increasing function.

For seller's utility indifference price the translation invariance property is evident since function $F_u(g, p_T^s) = 1 + \frac{p_T^s}{x} - \frac{g(u)}{x}$ has a property

$$F_u(g + m, p_T^s - m) = F_u(g, p_T^b), \quad \forall m \in \mathbb{R}.$$

The convexity of $p_T^s(g)$ can be deduced from the following. Let us consider that $g^{(1)}, g^{(2)}$ are $\mathcal{B}(\mathbb{R}_+)$-measurable functions and $p_T^{(1)}, p_T^{(2)}$ are the corresponding sellers's indifference prices. Then,

$$\int_{\mathbb{R}} \ln\left(1 + \frac{\alpha p_T^{(1)} - (1-\alpha)p_T^{(2)}}{x} + \frac{\alpha g_1(u) - (1-\alpha)g_2(u)}{x}\right) d\mu(u)$$
$$\leq \alpha \int_{\mathbb{R}} \ln\left(1 + \frac{p_T^{(1)}}{x} - \frac{g_1(u)}{x}\right) d\mu(u) + (1-\alpha) \int_{\mathbb{R}} \ln\left(1 + \frac{p_T^{(2)}}{x} - \frac{g_2(u)}{x}\right) d\mu(u),$$

and it follows that

$$p_T^b(\alpha g_1 + (1-\alpha)g_2) \leq \alpha p_T^b(g_1) + (1-\alpha)p_T^b(g_2),$$

and $p_T^s(g)$ is a convex functional. Finally, let $g_1(u) \leq g_2(u)$, $u \in \mathbb{R}$. Then we have for the seller's prices $p_T^s(g_1)$ and $p_T^s(g_2)$ that

$$\int_{\mathbb{R}} \ln\left(1 + \frac{p_T^s(g_1)}{x} - \frac{g_1(u)}{x}\right) d\mu(u) \geq \int_{\mathbb{R}} \ln\left(1 + \frac{p_T^s(g_1)}{x} - \frac{g_2(u)}{x}\right) d\mu(u).$$

Since the left hand side of the inequality is equal to 0, and the function

$$F(y) = \int_{\mathbb{R}} \ln\left(1 + \frac{y}{x} - \frac{g_2(u)}{x}\right) d\mu(u)$$

is increasing function, we conclude that $p_T^s(g_1) \geq p_T^s(g_2)$,

The concavity, monotonicity and translation invariance property of the buyer's indifference price p_T^b can be proved using the same arguments as for p_T^s.

□

Proposition 10. *Let $V(x,g)$ is defined by* (11) *and let the conditions of Proposition 7 are satisfied. If $\int_{\mathbb{R}} (x - g(u))^p \, d\mu < \infty$, $\int_{\mathbb{R}} (g(u))^p d\mu < \infty$ for $p < 0$ and $g(u) \in (0, x)$, $(\mu - a.s.)$, then the buyer's and seller's indifference prices for the power utility $U(x) = \frac{x^p}{p}$, $p \in (-\infty, 0) \cup (0, 1)$ are respectively defined from*

$$\int_{\mathbb{R}} \left[\left(1 - \frac{p_T^b}{x} + \frac{g(u)}{x}\right)^p - 1\right] d\mu(u) = 0 \tag{71}$$

and

$$\int_{\mathbb{R}} \left[\left(1 + \frac{p_T^s}{x} - \frac{g(u)}{x}\right)^p - 1\right] d\mu(u) = 0. \tag{72}$$

The equations (71) *and* (72) *have the unique solutions.*

Moreover, $p_T^b(g)$ is concave, increasing functional and $p_T^s(g)$ is convex, decreasing functional, which both satisfy the translation invariance property: $\forall\ m \in \mathbb{R}$

$$p_T^{b,s}(g + m) = p_T^{b,s}(g) - m.$$

Proof:

To solve the indifference pricing problems (67) and (68) when the investor's preferences are described by power utility we use Proposition 8. The values of the left hand sides of (67) and (68) coincide and can be obtained by taking $g = 0$ in (66). To calculate the right hand side of (67) we substitute x by $x - p_T^b$. The value function on the right hand side of (68) corresponds to the situation of the selling the option and, hence, g is replaced by $-g$ and x is substituted by $x + p_T^s$ in (66). Therefore, from (67) and (66) we have

$$\int_{\mathbb{R}} \left(\frac{x - p_T^b + g(u)}{H_T^{(q*)}}\right)^p H_T^{(q*)} d\mu(u) = \int_{\mathbb{R}} \left(\frac{x}{H_T^{(q*)}}\right)^p H_T^{(q*)} d\mu(u).$$

This equation is equivalent to (71). Formula (72) can be obtained from the relation

$$p_T^b(g) = -p_T^s(-g).$$

Let us denote by $F(y)$ the function on $[0, x]$ of the form:

$$F(y) = \int_{\mathbb{R}} \left[\left(1 - \frac{y}{x} + \frac{g(u)}{x} \right)^p - 1 \right] d\mu(u), \quad y \in [0, x]\,.$$

The function $\left(1 - \frac{y}{x} + \frac{g(u)}{x}\right)^p$ is a well-defined decreasing for $p \in (0, 1)$ (increasing for $p < 0$) function of y on $[0, x]$ if $g(u) \in (0, x)$. Hence,

$$\frac{g(u)}{x} \le 1 - \frac{y}{x} + \frac{g(u)}{x} \le 1 + \frac{g(u)}{x},$$

$$\left(\frac{g(u)}{x}\right)^p \le \left([1 - \frac{y}{x} + \frac{g(u)}{x}\right)^p \le \left(1 + \frac{g(u)}{x}\right)^p, \quad p \in (0, 1),$$

$$\left(\frac{g(u)}{x}\right)^p \ge \left([1 - \frac{y}{x} + \frac{g(u)}{x}\right)^p \ge \left(1 + \frac{g(u)}{x}\right)^p, \quad p < 0.$$

We can verify that under the condition $\int_{\mathbb{R}} (g(u))^p \, d\mu(u) < \infty$, for $p < 0$, the function $F(y)$ is a well-defined continuous function on $[0, x]$ for $p \in (-\infty, 0) \cup (0, 1)$.

Since $F(0) \ge 0$, $0 \le F(0) < \infty$ for $p \in (0, 1)$ and $F(0) \le 0$, $0 \le F(x) < \infty$ for $p < 0$, then, by the mean-value theorem there exists a solution on $[0, x]$ for (71) .

Next, we prove the uniqueness of solution of the equation $F(y) = 0$. The uniqueness follows from the fact that $F(y)$ is a strictly decreasing function if $p \in (0, 1)$ and is a strictly increasing if $p < 0$.

We denote by $\tilde{f}(y, u)$, $(y, u) \in [0, x] \times \mathbb{R}$ the integrable with respect to μ function of the form:

$$\tilde{f}(y, u) = \left(1 - \frac{y}{x} + \frac{g(u)}{x}\right)^p - 1.$$

The function $\tilde{f}(y, u)$ is continuous in y and u on $[0, x] \times \mathbb{R}$ and its derivative

$$\tilde{f}_y(y, u) = -\frac{p}{x} \left(1 - \frac{y}{x} + \frac{g(u)}{x}\right)^{p-1}$$

is also continuous on $[0, x] \times \mathbb{R}$.

In fact, if $g(u) \in (0,x)$, then $1-\frac{y}{x}+\frac{g(u)}{x}>0$ and for $p\in(0,1)$, $\tilde{f}_y(y,u)<0$ and for $p<0$, $\tilde{f}_y(y,u)>0$ for all $(y,u)\in[0,x]\times\mathbb{R}$. Moreover, for all $p\in(0,1)$, $x>0$, $(y,u)\in[0,x]\times\mathbb{R}$:

$$\tilde{f}_y(y,u) < -\frac{p}{x}2^{p-1} < 0,$$

and for all $p<0$, $x>0$, $(y,u)\in[0,x]\times\mathbb{R}$:

$$\tilde{f}_y(y,u) > -\frac{p}{x}2^{p-1} > 0.$$

Then,

$$\frac{\partial}{\partial y}F(y) = \int_{\mathbb{R}} \tilde{f}_y(y,u)d\mu(u)$$

and for $p\in(0,1)$:

$$\frac{\partial}{\partial y}F(y) \le -\frac{p}{x}2^{p-1} < 0,$$

and for $p<0$:

$$\frac{\partial}{\partial y}F(y) \ge -\frac{p}{x}2^{p-1} > 0.$$

Therefore, $F(y)$ is a strictly decreasing function for $p\in(0,1)$ and is a strictly increasing function for $p<0$ on $[0,x]$.

In the case of the seller's indifference price, we denote

$$F(y) = \int_{\mathbb{R}} \left[\left(1+\frac{y}{x}-\frac{g(u)}{x}\right)^p - 1\right] d\mu(u), \quad y\in[0,x]\,.$$

We see that $F(y)$ is increasing function for $p\in(0,1)$ and decreasing function for $p<0$ on $[0,x]$ for $g(u)\in(0,x)$. If $\int_{\mathbb{R}}(x-g(u))^p\,d\mu<\infty$ for $p<0$, then the function $F(y)$ is well defined continuous function on $[0,x]$ for $p\in(-\infty,0)\cup(0,1)$. Since $F(0)\le 0$, $0\le F(x)<\infty$ for $p\in(0,1)$ and $F(x)\le 0$, $0\le F(0)<\infty$ for $p<0$, then, by the same arguments as in the case of buyer's price, the equation (72) has a a unique solution on $[0,x]$.

For seller's utility indifference price the translation invariance property is justified by the fact that function $F_u(g,p_T^s)=1+\frac{p_T^s}{x}-\frac{g(u)}{x}$ has a property

$$F_u(g+m,p_T^s-m)=F_u(g,p_T^b),\quad \forall m\in\mathbb{R}.$$

Let us take $g^{(1)},g^{(2)}$ which are $\mathcal{B}(\mathbb{R}_+)$-measurable functions and let $p_T^{(1)},p_T^{(2)}$ be the corresponding sellers's indifference prices. If $p(p-1)>0$, then

$$\int_{\mathbb{R}}\left(\left(1+\frac{\alpha p_T^{(1)}-(1-\alpha)p_T^{(2)}}{x}+\frac{\alpha g_1(u)-(1-\alpha)g_2(u)}{x}\right)^p-1\right)d\mu(u)$$
$$\leq \alpha\int_{\mathbb{R}}\left(\left(1+\frac{p_T^{(1)}}{x}-\frac{g_1(u)}{x}\right)^p-1\right)d\mu(u)+(1-\alpha)\int_{\mathbb{R}}\left(\left(1+\frac{p_T^{(2)}}{x}-\frac{g_2(u)}{x}\right)^p-1\right)d\mu(u),$$

and it follows that

$$p_T^b(\alpha g_1+(1-\alpha)g_2)\leq \alpha p_T^b(g_1)+(1-\alpha)p_T^b(g_2),$$

and $p_T^s(g)$ is a convex functional.

Let $g_1(u)\leq g_2(u)$, $u\in\mathbb{R}$. Then we have for the sellers's prices $p_T^s(g_1)$ and $p_T^s(g_2)$ that

$$\int_{\mathbb{R}}\left(\left(1+\frac{p_T^s(g_1)}{x}-\frac{g_1(u)}{x}\right)^p-1\right)d\mu(u)\geq\int_{\mathbb{R}}\left(\left(1+\frac{p_T^s(g_1)}{x}-\frac{g_2(u)}{x}\right)^p-1\right)d\mu(u).$$

Since the left hand side of the inequality is equal to 0 and

$$F(y)=\int_{\mathbb{R}}\left(\left(1+\frac{y}{x}-\frac{g_2(u)}{x}\right)^p-1\right)d\mu(u)$$

is increasing function, we conclude that $p_T^s(g_1)\geq p_T^s(g_2)$.

The concavity and monotonicity of the buyer's indifference price p_T^b can be proved using the same arguments as for p_T^s. The case of $p\in(0,1)$ can be considered in the similar way.

□

Proposition 11. *Let $V(x,g)$ is defined by* (11) *and let the conditions of Proposition 7 are satisfied. If $V(x,0)<\infty$ and $V(x,g)<\infty$, then the buyer's and seller's indifference prices in the case of the exponential utility $U(x)=1-e^{-\gamma x}$, $\gamma>0$ are given by*

$$p_T^b=-\frac{1}{\gamma}\ln\left[\int_{\mathbb{R}}\exp\{-\gamma g(u)\}d\mu\right] \tag{73}$$

and

$$p_T^s=\frac{1}{\gamma}\ln\left[\int_{\mathbb{R}}\exp\{\gamma g(u)\}d\mu\right]. \tag{74}$$

Moreover, $p_T^b(g)$ is concave, monotone increasing functional and $p_T^s(g)$ is convex, monotone decreasing functional, which both satisfy the translation invariance property: $\forall\ m \in \mathbb{R}$

$$p_T^{b,s}(g+m) = p_T^{b,s}(g) - m.$$

Proof:

In the case of the exponential utility $-f^{'}(y) = -\frac{1}{\gamma}(\ln y - \ln \gamma)$, and we have

$$\begin{aligned} V_T(x - p_T^b, g) &= \int_{\mathbb{R}} E_P\left[U\left(-f^{'}\left(\tilde{\lambda}^*(u) Z_T^*\right)\right)\right] d\mu(u) \\ &= 1 - \frac{1}{\gamma}\int_{\mathbb{R}} \tilde{\lambda}^*(u) d\mu(u) \end{aligned}$$

and

$$V_T(x,0) = 1 - \frac{\tilde{\lambda}^*}{\gamma},$$

where $\tilde{\lambda}^*(u)$ is given by (56) and $\tilde{\lambda}^*$ can be obtain by putting $g = 0$ in (56).

Thus, taking into account that $E_P\left[Z_T^* \ln Z_T^*\right] = \mathbf{I}(Q_T^*|P_T)$ we obtain the following formula for the indifference price p_T^b in the case of the exponential utility:

$$\int_{\mathbb{R}} \exp\left\{-\gamma\left(x - p_T^b + g(u)\right) - \mathbf{I}(Q_T^*|P_T)\right\} d\mu(u) = \exp\left\{-\gamma x - \mathbf{I}(Q_T^*|P_T)\right\}.$$

This equation is equivalent to (73) and the seller's indifference price (74) we can obtain using relation

$$p_T^b(g) = -p_T^s(-g).$$

Evidently, the translation invariance and monotonicity properties are verified for the buyer's and seller's exponential indifference prices. Let us take $g(u) = \alpha g_1(u) + (1-\alpha) g_2(u)$, $\alpha \in (0,1)$, then by Holder inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$ we obtain:

$$\begin{aligned} \int_{\mathbb{R}} \exp\{\gamma g(u)\} d\mu(u) \leq \left(\int_{\mathbb{R}} \exp\{\gamma g_1(u)\} d\mu(u)\right)^{\alpha} \\ \times \left(\int_{\mathbb{R}} \exp\{\gamma g_2(u)\} d\mu(u)\right)^{1-\alpha}, \end{aligned}$$

that proves the convexity of p_T^s defined by (74). The concavity of p_T^b defined by (73) can be proved using the same arguments.

□

Remark 1. *From Proposition 9, Proposition 10 and Proposition 11, we conclude that in the case when the Levy process L and random variable ξ are independent, the formulae of the logarithmic, exponential, and power utility indifference prices are independent on the available strong insider information about a random variable ξ on the considered market. (For the details see Section 2).*

5. Application to defaultable Geometric Brownian Motion model

We fix T as maturity time of the option. Let $(\Omega, \mathcal{F}_T P)$ be a probability space on which we define two-dimensional Brownian motion $W = (W_t^{(1)}, W^{(2)})_{0\le t\le T}$. We endow $(\Omega, \mathcal{F}_T, P)$ with a filtration $\mathbb{F}^{(1)} = (\mathcal{F}_t^{(1)})_{0\le t\le T}$ generated by $W^{(1)}$.

We consider the model which supports one traded asset $S^{(1)}$ and defaultable bond $S^{(2)}$. The assets $S^{(1)}$ and $S^{(2)}$ are two independent geometric Brownian motions such that

$$S_t^{(1)} = S_0^{(1)} \exp\left\{ \left(\mu_{(1)} - \frac{\sigma_{(1)}^2}{2}\right) t + \sigma_{(1)} W_t^{(1)} \right\}$$

$$S_t^{(2)} = S_0^{(2)} \exp\left\{ \left(\mu_{(2)} - \frac{\sigma_{(2)}^2}{2}\right) t + \sigma_{(2)} W_t^{(2)} \right\},$$

where $\mu^{(\cdot)}$ and $\sigma^{(\cdot)}$ are the drift and diffusion coefficients respectively and $W^{(\cdot)}$ is a Wiener process.

Finally, we define a random variable τ as a default time, i.e. a first time when the stock price process $S^{(2)}$ hits a barrier $a \in (0,1)$:

$$\tau = \inf\left\{t \in [0,T] : S_t^2 \le a\right\}.$$

Then,the distribution of τ is

$$F_\tau(t) = \Phi\left(\frac{\ln a - (\mu_{(2)} - \frac{\sigma_{(2)}^2}{2})t}{\sigma_{(2)}\sqrt{t}}\right) + a^{2\frac{\mu_{(2)}}{\sigma_{(2)}^2} - 1} \Phi\left(\frac{\ln a + (\mu_{(2)} - \frac{\sigma_{(2)}^2}{2})t}{\sigma_{(2)}\sqrt{t}}\right).$$

Then the information available to the insider at time t is represented by sigma-algebra $\mathcal{G}_t = \bigcap_{s>t}\left(\mathcal{F}_t^{(1)} \otimes \sigma(\tau)\right)$. (For more details about the model and the distribution see [28])

We suppose that investor who buys the bond will receive a payment b at time T if and only if default has occurred before time T, i.e. the payoff function of the bond is $g\left(\mathbb{I}_{\{\tau\le T\}}\right) = b\mathbb{I}_{\{\tau\le T\}}$. For simplicity of calculations we assume that $S_0^{(\cdot)} = 1$, $\mu_{(\cdot)} = 0$ and $\sigma_{(\cdot)} = 1$, then the distribution of τ is

$$F_\tau(t) = \Phi\left(\frac{\ln a + \frac{t}{2}}{\sqrt{t}}\right) + \frac{1}{a}\Phi\left(\frac{\ln a - \frac{t}{2}}{\sqrt{t}}\right). \tag{75}$$

We assume the initial capital x be equal to 1. Then according to Proposition 8 and Proposition 9, for the existence of solutions in the equations (69), (70), (71) and (72), in the case of the logarithmical and power utilities the monetary value b must not exceed 1.

Next, using the formula (75) we calculate the distributions of τ in the different cases of the barrie a and for different maturity time T.

Table 1. Distribution of τ $F_\tau(T)$

Case	$T = 1$	$T = 1.5$	$T = 2$	$T = 2.5$	$T = 3$
$a = 0.1$	0.06107412‘	0.16589305‘	0.27615169‘	0.37604460‘	0.46221476‘
$a = 0.2$	0.22088765‘	0.37653772‘	0.49579569‘	0.58641865‘	0.65635072‘
$a = 0.3$	0.38803513‘	0.53980954‘	0.64120586‘	0.71270390‘	0.76533803‘
$a = 0.4$	0.53446163‘	0.66308077‘	0.74286328‘	0.79690473‘	0.83569574‘
$a = 0.5$	0.65623355‘	0.7571794‘	0.81710178‘	0.85673907‘	0.88477023‘

5.1. **Exponential indifference prices.** To calculate the exponential buyer's indifference price we use the formula (73) from Proposition 11:

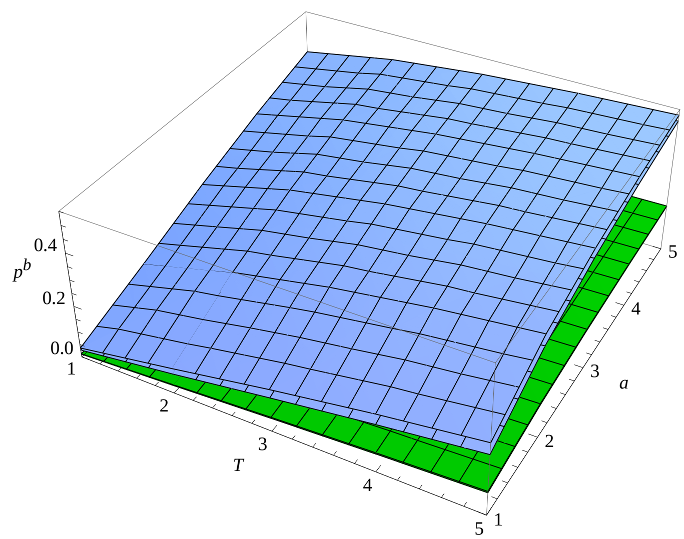
$$p_T^b = -\frac{1}{\gamma}\ln\left[\int_{\mathbb{R}} \exp\left\{-\gamma g\left(u\right)\right\} d\mu(u)\right],$$

where $\gamma > 0$.

Then, in the situation of the considered defaultable market driven by the geometric Brownian motion, where the random variable τ is defined on $\left([0,T], \mathcal{B}\left([0,T]\right)\right)$, we obtain the following expression for the exponential indifference price:

$$\begin{aligned}
p_T^b &= -\frac{1}{\gamma}\ln\left[\int_0^T \exp\left\{-\gamma g\left(\mathbb{I}_{\{\tau\le T\}}\right)\right\} dP_{\{\tau\le T\}}\right] \\
&= -\frac{1}{\gamma}\ln\left[\int_0^T \exp\{-\gamma b\} dP_{\{\tau\le T\}} + \int_0^T dP_{\{\tau> T\}}\right] \\
&= -\frac{1}{\gamma}\ln\left[\exp\{-\gamma b\}F_\tau(T) + 1 - F_\tau(T)\right].
\end{aligned}$$

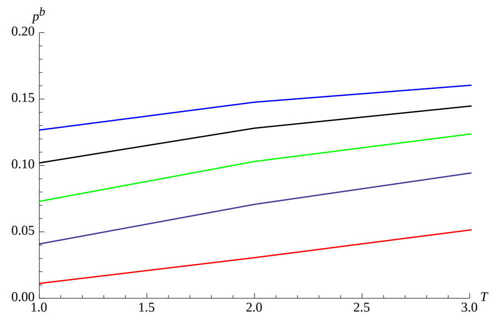
Then, we can calculate the exponential buyer's indifference prices. For comparison, we take two different premiums $b = 0.2$ and $b = 0.6$, which investor receive in the case if default has occurred before time T. We use the graph to demonstrate the result for exponential buyer's indifference prices plotted for different initial data.



The exponential utility indifference prices for $\gamma \in (0.1, 2)$.

The corresponding values of the axes T and a are from the grid $[5 \times 5]$ of Table 1.The dark sheets corresponds to the case of $b = 0.2$ and the light sheets to $b = 0.6$. The different layers of the sheets correspond to the different coefficient of risk aversion $\gamma > 0$.

On the next graph we observe the section of the previous graph taken with respect to different values of the hitting barrier a.



The exponential utility indifference prices in the case $\gamma = 1$.

The lines on the graph from the bottom to the top correspond to the values of a equal to 0.1, 0.2, 0.3, 0.4 and 0.5 respectively.

From these two graphs we can conclude that the exponential buyer's indifference prices increase with the growth of the value of the hitting barrier a and with the growth of the terminal time T.

5.2. **Numerical result for the logarithmic and power indifference prices.** To obtain the equation for the logarithmic buyer's indifference price we use the equation (69) from Proposition 9:

$$\int_{\mathbb{R}} \ln\left[1 - \frac{p_T^b}{x} + \frac{g(u)}{x}\right] d\mu(u) = 0.$$

Then, in the situation of the considered defaultable market driven by the geometric Brownian motion, where the random variable τ is defined on $([0,T], \mathcal{B}([0,T]))$ and $x = 1$, using (69) we get:

$$\begin{aligned} 0 &= \int_0^T \ln\left[1 - p_T^b + g\left(\mathbb{I}_{\{\tau\le T\}}\right)\right] dP_{\{\tau\le T\}} \\ &= \int_0^T \ln\left[1 - p_T^b + b\right] dP_{\{\tau\le T\}} + \int_0^T \ln\left[1 - p_T^b\right] dP_{\{\tau> T\}} \\ &= \ln\left[1 - p_T^b + k\right] F_\tau(T) + \ln\left[1 - p_T^b\right](1 - F_\tau(T)). \end{aligned}$$

Thus, the logarithmic buyer's indifference price p_T^b is a solution of the equation

$$\ln\left[1 - p_T^b + b\right] F_\tau(T) + \ln\left[1 - p_T^b\right](1 - F_\tau(T)) = 0. \tag{76}$$

For the power byuer's indifference price in the case $p<1, p\neq 0$ we use the equation (71) from Proposition 9 and we get:

$$\begin{aligned} 0 &= \int_0^T \left[\left(1-p_T^b+g\left(\mathbb{I}_{\{\tau\leq T\}}\right)\right)^p-1\right] dP_{\{\tau\leq T\}} \\ &= \int_0^T \left[\left(1-p_T^b+b\right)^p-1\right] dP_{\{\tau\leq T\}}+\int_0^T \left[\left(1-p_T^b\right)^p-1\right] dP_{\{\tau> T\}} \\ &= \left(\left(1-p_T^b+k\right)^p-1\right) F_\tau(T)-\frac{1}{2}+\left(\left(1-p_T^b\right)^p-1\right)\left(1-F_\tau(T)\right). \end{aligned}$$

Then the power buyer's indifference price p_T^b is the solution of the equation

$$\left(\left(1-p_T^b+k\right)^p-1\right) F_\tau(T)-\frac{1}{2}+\left(\left(1-p_T^b\right)^p-1\right)\left(1-F_\tau(T)\right)=0. \tag{77}$$

We have only the numerical result for the equations (76) and (77). The following numerical result was obtained with using of the command NSolve in MathematicaWolfram. The logarithmic buyer's indifference prices and the power ($p=-1/2$ and $p=1/2$) buyer's indifference prices calculated for the hitting barrier $a=0.1$, the premium $b=0.2$ and the different maturity times T are shown in the Table 2. The corresponding values of the exponential buyer's indifference prices are displayed for the comparison.

TABLE 2. Buyer's indifference prices

Case $a=0.1,\ b=0.2$	$p_T^{b,exp},\ \gamma=1$	$p_T^{b,log}$	$p_T^{b,1/2}$	$p_T^{b,-1/2}$
$T=1$	0.0111326	0.0111871	0.0107143	0.00984339
$T=1.5$	0.0305353	0.0306383	0.0294511	0.0272343
$T=2$	0.0513541	0.0514628	0.0496708	0.0462718
$T=2.5$	0.0705999	0.0706757	0.0684823	0.0642565
$T=3$	0.0875046	0.087533	0.0851196	0.0804011

We can see from Table 2 that for our example of the defaultable geometric brownian motion model, the highest buyer's indifference prices was obtained in the case when the preferences of the investor was modelled with using of logarithmic utility. In the case of the exponential utility, we considered the risk aversion coefficient equal to 1, however the more risk-averse investor with $\gamma<1$ will pay higher indifference price and the risk-loving investor with $\gamma>1$ will pay less. In the case of power utility, from right hand side of formula (71) we can see that the power indifference price p_T^b increases when the power p decreases in the case $p \in (0, 1)$ and vice versa in the case $p < 0$.

However, in our example, the relative difference between the considered types of the utility indifference prices not grater than 9% (Table 2). The analysis of the results over other types of utilities gives that the relative difference between the indifference prices will highly increase with the increasing of the risk averse factor γ and with decreasing of $p < 0$.

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